The Arching of Beams

By

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Introduction:

The ancient problem of curved beams arching, explored by the Greeks and Romans, is applied, in this paper, to a straight beam instead of an arch. The following synopsis offers a new insight into the phenomenon of a straight beam behaving as an arch. Using finite element (FE), one observes the arching action when looking at the load distribution of a stress diagram. How, though, can FE be used to find the reductions in moment and shear due to the redistribution of the dead and live loads caused by arching? This dilemma is especially apparent in the analysis of concrete because FE does not accept reinforcing rebar; instead, 3-D analysis is needed. Additionally, modeling becomes very cumbersome if FE analysis must be performed for every beam in a building. It would be much more productive if one knew the arching factors beforehand. Elasticity will allow the exploration of this phenomenon in all materials, including soils (for small strains).

In a previous paper, the problem was solved with an infinite soil or material on top of a beam in one bay, or infinite bay (see http://www.facsystems.com/LAGGING.pdf). If bricks are substituted for soil, steel channel for wood lagging, and columns for soldier piles, then it can be concluded that more arching reduction would result as the columns become thicker. Additionally, the shear reduction factor is different than the bending moment reduction factor. The solution in that paper is very precise with less than 2% difference between the results calculated using the reduction factors and those calculated directly. Utilizing the substitutions above, we will explore a confined soil or material, on top of a beam, of which the material ends at a distance *h* above the beam. We will also assume a live load on top of the material. Future research will explore the arching of materials without a beam at the bottom, including wide flange steel beams.

This paper represents the latest iteration of a thought process initiated during my Master's Thesis in which I proposed a correction to an error by W. D. Liam Finn, PhD, in his paper, "Boundary Value Problems of Soil Mechanics." My goal was to modify his method to prove the existence of arching. The conclusion was discovering the arching factors for wood lagging in shoring walls. There was an error in differentiating the deflection, however, because the deflection, expressed as a Fourier series, was calculated by taking the fourth derivative of the series. This represented an error in the boundary condition because differentiating a Fourier series is not allowed. Furthermore, ignoring the load on top of the pile by taking a Fourier series on the lagging only was incorrect because the load on top of the pile affects the arching and should be considered. Regardless of the error, though, the Master's Thesis mathematically proves the existence of arching.

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The thesis error was corrected in the article cited above (facsystems link). It is evident from the collocation function method how to arrive at the solution by integrating the stress to equate the deflection. What is not clear, however, is how to utilize the Fourier series representation to obtain the exact deflection. Further research is needed. In the foregoing analysis, the exact solution, if found, would be easier to implement than the collocation function.

This paper, for which the master's thesis research and the web article paved the road, will evaluate the infinite bay solution instead of the one bay solution, which are similar when using the Fourier Transform instead of the Fourier series.

Keeping in mind that if there is no deflection, as in rigid beam, then there is no arching and the full load must be taken into account, we start with the phi function in elasticity:

$$\Phi = \sum_{n=\phi}^{\infty} \frac{1}{\alpha_n^2} \cos \alpha_n x \left[(A + B\alpha_n y) e^{\alpha_n y} + (C + D\alpha_n y) e^{-\alpha_n y} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 1$$

Where x and y are the coordinates axis and A, B, C and D are constant and at $n = \phi$ is at $n \to 0$ but divide by 2 as in the Fourier cosine series first term. α_n is a function of n, $\alpha_n = \dfrac{n\pi}{b+t}$. Where $2b$ is the bay width and $2t$ is the column width as in Fig. 1. Thus the stress is

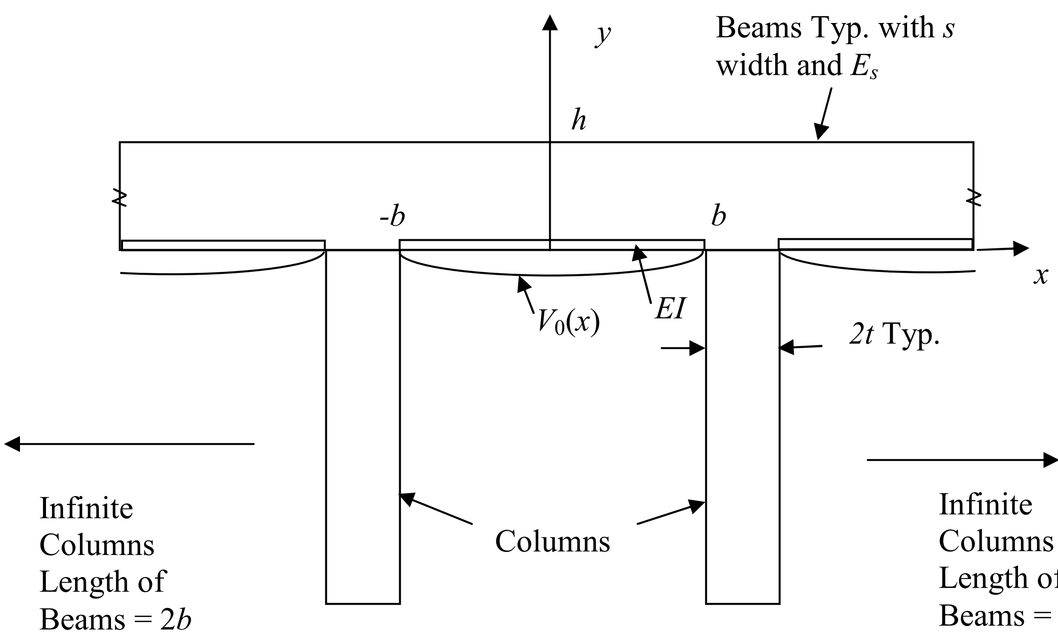


Fig. 1 Beams and Columns Diagram With Bottom Beam

$$\sigma_y = \frac{\partial^2 \Phi}{\partial x^2} = -\sum_{n=\phi}^{\infty} \cos\alpha_n x\left[(A + B\alpha_n y)e^{\alpha_n y} + (C + D\alpha_n y)e^{-\alpha_n y}\right] \quad \ldots\ldots 2$$

At $y = h$ there is the live load $-q$ expressed as a Fourier series = $\sum_{n=\phi}^{\infty} w_n \cos\alpha_n x$ ……………… 3

Thus:

$$(A + B\alpha_n h)e^{\alpha_n h} + (C + D\alpha_n h)e^{-\alpha_n h} = -w_n \quad \ldots\ldots 4$$

Or

$$A = -(C + D\alpha_n h)e^{-2\alpha_n h} - B\alpha_n h - w_n e^{-\alpha_n h} \quad \ldots\ldots 5$$

Now

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \sum_{n=\phi}^{\infty} \sin\alpha_n x\left[(A + B + B\alpha_n y)e^{\alpha_n y} + (-C + D - D\alpha_n y)e^{-\alpha_n y}\right] \quad \ldots\ldots 6$$

We will assume the shear is zero at $y = h$, thus from Eq. 6 yields:

$$(A + B + B\alpha_n h)e^{\alpha_n h} + (-C + D - D\alpha_n h)e^{-\alpha_n h} = 0 \quad \ldots\ldots 7$$

Substituting Eq. 5 in Eq. 7 and solve for B yields:

$$B = (2C - D + 2D\alpha_n h)e^{-2\alpha_n h} + w_n e^{-\alpha_n h} \quad \ldots\ldots 8$$

Substitute Eq. 8 in Eq. 5 yields:

$$A = -\left(C + 2C\alpha_n h + 2D\alpha_n^2 h^2\right)e^{-2\alpha_n h} - w_n(\alpha_n h + 1)e^{-\alpha_n h} \quad \ldots\ldots 9$$

We will assume at $y = 0$ the shear is zero, thus from Eq. 6 yields

$$A + B - C + D = 0 \quad \ldots\ldots 10$$

Substitute Eq. 8 and Eq. 9 in Eq. 10 and solve for C yields:

$$C = \frac{-1 + \left(1 - 2\alpha_n h + 2\alpha_n^2 h^2\right)e^{-2\alpha_n h}}{(1 - 2\alpha_n h)e^{-2\alpha_n h} - 1} D + \frac{\alpha_n h e^{-\alpha_n h}}{(1 - 2\alpha_n h)e^{-2\alpha_n h} - 1} w_n \quad \ldots\ldots 11$$

Let

$$C = c_n D + w_{n3} w_n \quad \ldots\ldots 12$$

Where

$$c_n = \frac{-1 + \left(1 - 2\alpha_n h + 2\alpha_n^2 h^2\right)e^{-2\alpha_n h}}{(1 - 2\alpha_n h)e^{-2\alpha_n h} - 1}$$

And

$$w_{n3} = \frac{\alpha_n h e^{-\alpha_n h}}{(1 - 2\alpha_n h)e^{-2\alpha_n h} - 1}$$

Substitute Eq. 12 in Eq. 9 and solve for *A* yields:

$$A = a_n D + w_{n1} w_n \quad \text{..} \quad 13$$

Where

$$a_n = -\left[c_n (1 + 2\alpha_n h) + 2\alpha_n^2 h^2\right]e^{-2\alpha_n h}$$

And

$$w_{n1} = -w_{n3}(1 + 2\alpha_n h)e^{-2\alpha_n h} - (1 + \alpha_n h)e^{-\alpha_n h}$$

Substitute Eq. 12 in Eq. 8 yields:

$$B = b_n D + w_{n2} w_n \quad \text{..} \quad 14$$

Where

$$b_n = \left[2c_n - 1 + 2\alpha_n h\right]e^{-2\alpha_n h}$$

And

$$w_{n2} = 2w_{n3} e^{-2\alpha_n h} + e^{-\alpha_n h}$$

So *A*, *B* and *C* are functions of *D*. Now we seek to find *D*.

To find *D* we need to find the deflection, the strain in the *y* direction is:

$$\varepsilon_y = \beta\sigma_y - \rho\sigma_x \quad \text{...} \quad 15$$

Where $\beta = \dfrac{1-\nu^2}{E_s}$ and $\rho = \dfrac{\nu(1+\nu)}{E_s}$ for plain strain and $\beta = \dfrac{1}{E_s}$ and $\rho = \dfrac{\nu}{E_s}$ for plain stress, ν is Poison's ration and E_s is the elastic modulus of the material on top of the beam and

$$\sigma_y = \frac{\partial^2 \Phi}{\partial x^2} = -\sum_{n=\phi}^{\infty} \cos\alpha_n x\left[(A + B\alpha_n y)e^{\alpha_n y} + (C + D\alpha_n y)e^{-\alpha_n y}\right]$$

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} = \sum_{n=\phi}^{\infty} \cos\alpha_n x\left[(A + 2B + B\alpha_n y)e^{\alpha_n y} + (C - 2D + D\alpha_n y)e^{-\alpha_n y}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 16$$

Thus:

$$\varepsilon_y = [-\beta A - \rho(A + 2B)]e^{\alpha_n y} + (-\beta - \rho)By\alpha_n e^{\alpha_n y} + [-\beta C - \rho(C - 2D)]e^{-\alpha_n y} + (-\beta - \rho)Dy\alpha_n e^{-\alpha_n y}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 17$$

Note: the summation sign and $\cos\alpha_n x$ is not shown for simplicity.

But

$$\varepsilon_y = \frac{\partial V(x)}{\partial y}$$

Then

$$\begin{aligned} V(x) = {} & [-\beta A - \rho(A + 2B)]\frac{e^{\alpha_n y}}{\alpha_n} - (-\beta - \rho)B\frac{e^{\alpha_n y}}{\alpha_n} + (-\beta - \rho)Bye^{\alpha_n y} \\ & + [-\beta C - \rho(C - 2D)]\frac{e^{-\alpha_n y}}{-\alpha_n} - (-\beta - \rho)D\frac{e^{-\alpha_n y}}{\alpha_n} - (-\beta - \rho)Dye^{-\alpha_n y} + g(x) \end{aligned} \quad \ldots\ldots\ 18$$

At $y = h \to \infty$ and $w_n = 0$, $V(x) = 0$ thus $g(x) = 0$ Note: A, B de-solves the equation to zero.

At $y = 0$ we have:

$$V_0(x) = \sum_{n=\phi}^{\infty} \frac{-\beta(A - B - C - D) - \rho(A + B - C + D)}{\alpha_n}\cos\alpha_n x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 19$$

At $C = D$ and $A = 0$ and $B = 0$ at $h \to \infty$ we have:

$V_0(x) = \sum_{n=\phi}^{\infty} \frac{2\beta D}{\alpha_n} \cos\alpha_n x$ matching the equation in the article at http://www.facsystems.com/LAGGING.pdf.

Substitute Eq. 12, 13 and 14 in Eq. 19 yields:

$$V_0(x) = -\sum_{n=\phi}^{\infty} \frac{\cos\alpha_n x}{\alpha_n} \{[\beta(a_n - b_n - c_n - 1) + \rho(a_n + b_n - c_n + 1)]D + \beta(w_{n1} - w_{n2} - w_{n3})w_n + \rho(w_{n1} + w_{n2} - w_{n3})w_n\} \quad \ldots\ldots 20$$

$V_0(x) = \sum_{n=\phi}^{\infty} v_n \cos\alpha_n x$ where $v_n = \frac{2}{b+t}\int_0^b v_0(\lambda)\cos\alpha_n \lambda d\lambda$ and $v_0(\lambda) = V_0$ is the deflection

Thus

$$D = -\frac{\alpha_n v_n + \beta(w_{n1} - w_{n2} - w_{n3})w_n + \rho(w_{n1} + w_{n2} - w_{n3})w_n}{\beta(a_n - b_n - c_n - 1) + \rho(a_n + b_n - c_n + 1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 21$$

$\sigma_y @ y = 0$ from Eq. 2 yields:

$$\sigma_y = -\sum_{n=\phi}^{\infty} \cos\alpha_n x(A + C) = -\sum_{n=\phi}^{\infty} \cos\alpha_n x[(a_n + c_n)D + (w_{n1} + w_{n3})w_n] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. 22$$

Substitute Eq. 21 in 22 yields:

$$\sigma_y = -\sum_{n=\phi}^{\infty} \cos\alpha_n x(k_n \alpha_n v_n + W_n) - p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 23$$

Where:

$k_n = -\frac{a_n + c_n}{d_n}$ and

$$W_n = -\frac{(a_n + c_n)[\beta(w_{n1} - w_{n2} - w_{n3})w_n + \rho(w_{n1} + w_{n2} - w_{n3})w_n]}{d_n} + (w_{n1} + w_{n3})w_n$$

And

$$d_n = \beta(a_n - b_n - c_n - 1) + \rho(a_n + b_n - c_n + 1)$$

And p is the self weight.

Since at $y = 0$

$$\int_0^{b+t} \sigma_y dx = -p(b+t) - q(b+t)$$

Then the $n = 0$ first coefficient drops and n starts at 1 for k_n factor since it does not involve w_n. For example, if $q = 0$, then the equation must hold for all n. We will use a parabola collocation function

$\bar{v}_0(z) = -d_1\delta\,[1-(x/b)^2]$

where $\delta=5sb^4p/(24EI)$ for dead loads, and $\delta=5sb^4q/(24EI)$ for live loads. The charts will be for dead load only and live load only and d_1 is a dimensional constant and will vary with h. Thus,

$$v_n = -\frac{4d_1\delta}{b+t}\left(\frac{1}{b^2\alpha_n^3}\sin\alpha_n b - \frac{1}{b\alpha_n^2}\cos\alpha_n b\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 24$$

It is clear that at $n\rightarrow 0$ $v_n = -\frac{2}{3}d_1\delta b$ and at $n = 0$ in the left term of in Eq. 23 is zero and the first term drops. For the live load:

$$w_n = -\frac{2}{b+t}\int_0^{b+t-l} q\cos\alpha_n\lambda d\lambda = -\frac{2q}{(b+t)\alpha_n}\sin\alpha_n(b+t-l) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 25$$

$$w_0 = -\frac{q}{b+t}(b+t-l)$$

where l is a small negligible distance.

Thus, the shear is:

$$\int \sigma_y dx = -\sum_{n=1}^{\infty}\sin\alpha_n x\left(k_n v_n + \frac{W_n}{\alpha_n}\right) - px \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 26$$

Note: n starts at 1 since W_n as $\alpha_n \rightarrow \infty$ is a finite number. Thus, the shear goes to infinity at $n = 0$, but, since the shear is finite and equal to $p(b+t) + q(b+t)$, $n = 0$ drops. Conversely, the constant of integration will de-solve it to have the shear as an odd function.

The shear reduction for dead load is:

$$R_v = \sum_{n=1}^{\infty}\frac{\sin\alpha_n b}{pb}(k_n v_n) + 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 27$$

The shear reduction for live load is:

$$R_v = \sum_{n=1}^{\infty} \frac{\sin \alpha_n x}{qb} \left(k_n v_n + \frac{W_n}{\alpha_n} \right) \quad \ldots\ldots 28$$

The moment is

$$\iint \sigma_y dx = \sum_{n=1}^{\infty} \cos \alpha_n x \left(\frac{k_n v_n}{\alpha_n} + \frac{W_n}{\alpha_n^2} \right) - p \frac{x^2}{2} + C1 \quad \ldots\ldots 29$$

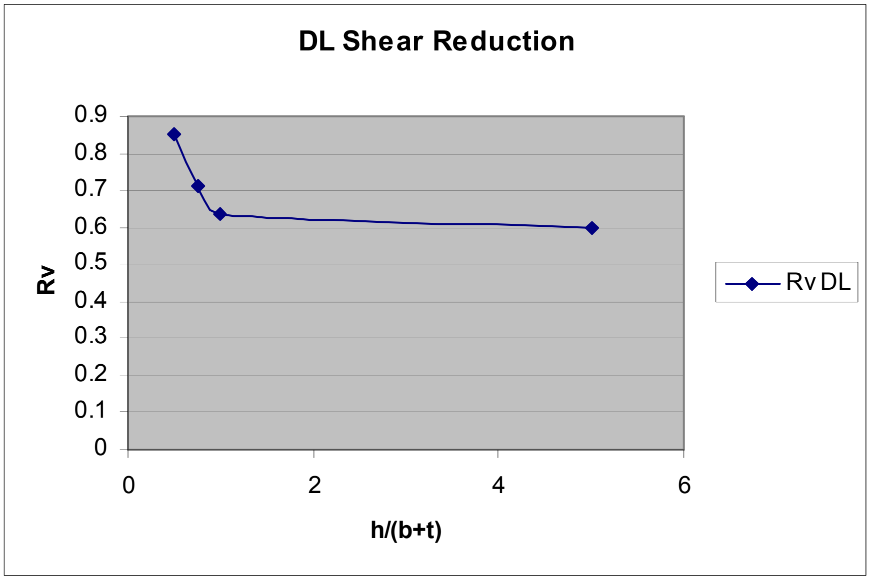
Where $C1$ is a constant of integration, the moment reduction for dead load is:

$$R_m = \sum_{n=1}^{\infty} \frac{(1 - \cos \alpha_n b)}{0.5 p b^2} \left(\frac{k_n v_n}{\alpha_n} \right) + 1 \quad \ldots\ldots 30$$

The moment reduction for live load is:

$$R_m = \sum_{n=1}^{\infty} \frac{(1 - \cos \alpha_n b)}{0.5 q b^2} \left(\frac{k_n v_n}{\alpha_n} + \frac{W_n}{\alpha_n^2} \right) \quad \ldots\ldots 31$$

See Fig 2 and 3 for arching reductions



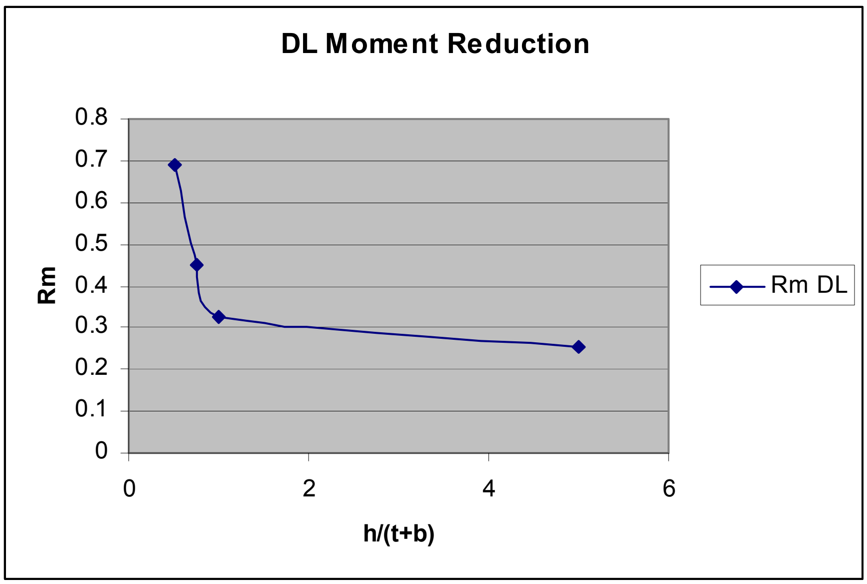


Figure 2 – Two Plots of the Reduction Factors due to Arching for DL

Note:

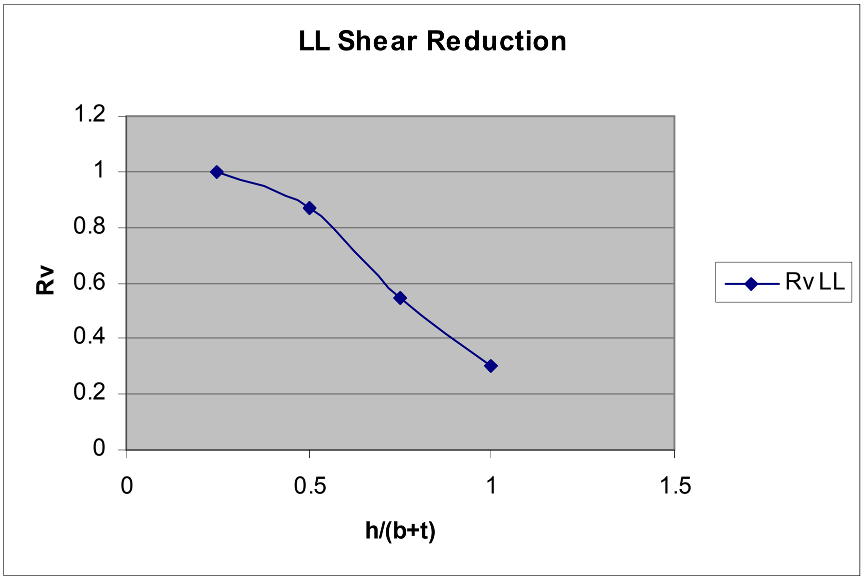
d1 = 0.282667

$$\frac{\delta}{\beta pb} = 6.25$$

b/(b+t) = 0.667

$\nu =$ 0.3 Soils

Note: h/(b+t) will change d1 and it is left constant for the purpose of demonstration
Charts are for demonstration only and not to be used until final charts
are derived from research



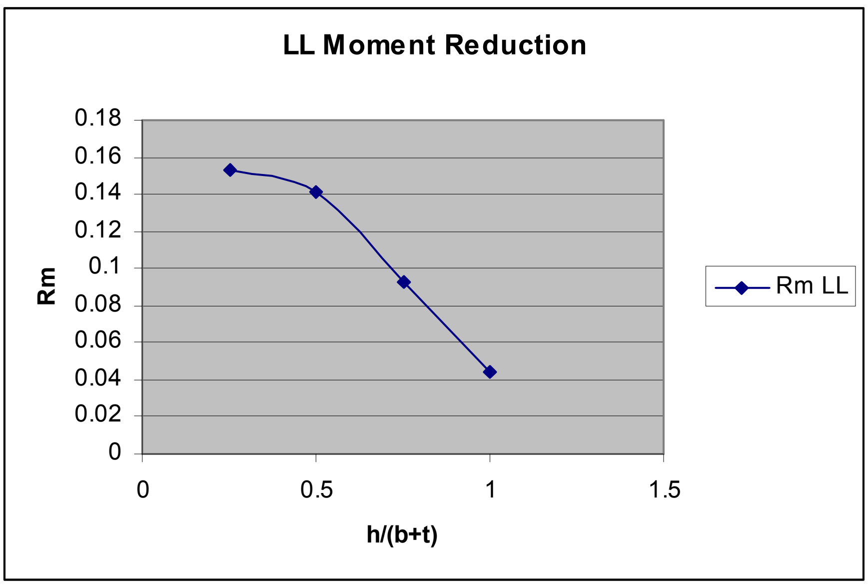


Figure 3 – Two Plots of the Reduction Factors due to Arching for LL

Note:

d1 = 0.01

$$\frac{\delta}{\beta q b} = 6.25$$

b/(b+t) = 0.667

$\nu =$ 0.3 Soils

(b+t-l)/(b+t)= 0.85

Note: h/(b+t) will change d1 and it is left constant for the purpose of demonstration
Charts are for demonstration only and not to be used until final charts are derived from research

Figure 4 shows the mechanics underling the loading.

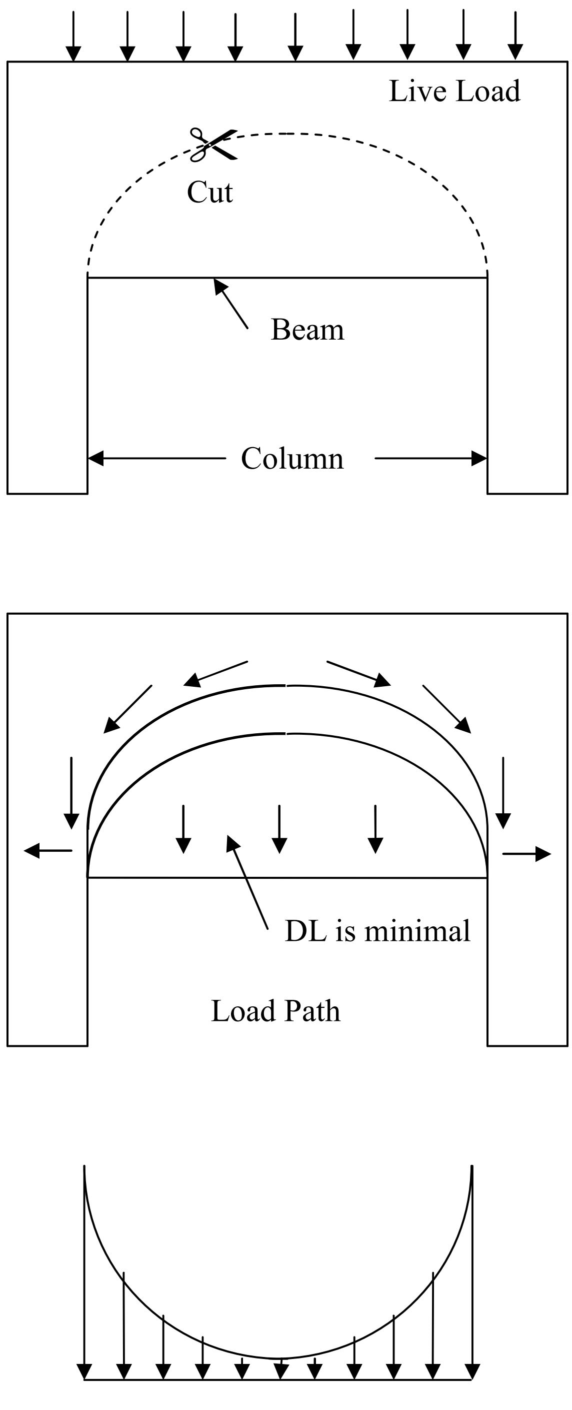


Figure 4 – When cut to an arch the load path and pressure diagram is exposed

Conclusion:

It is evident from the charts that *h* is proportional to the arching reduction obtained. This synopsis is worthy of further research and a candidate for a Ph. D. dissertation. The future research would involve systems' optimization with arching for any beam. The dividends of this research will produce great benefits for structural designs, not just for the Army Corps of Engineers, but for all Federal agencies, including the Forest Service, Aerospace Engineering (Air force), and Naval Architectures (Navy). The application in civil engineering includes culverts with ribs, tunneling with ribs, bridges with ribs, masonry buildings, concrete buildings, steel buildings and timber structures. The application in Aerospace and Naval Architectures is in reducing the weight of materials.

Reducing construction materials reduces energy consumption which reduces construction costs. In shoring and excavation systems, a 50% reduction from the empirical formula due to arching gives a 30% ($\sqrt{.5}$) reduction in materials (i.e. wood lagging over the entire face of the wall). This results in, approximately, a 15% reduction in total cost. Preliminary designs for deep culverts show 20% reduction in construction cost due to utilizing arching. If there is a reduction of 20% due to arching, as realized in the above charts for a moderate *h* for live or dead loads, then an average of 10% ($\sqrt{.8}$) reduction in materials can be achieved for most applications.