General Solution for Elastic Buckling and Ultimate Buckling for a Non-Prismatic Beams

by

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In this paper we use the finding in the paper by the author on "Exact and Numerical Solutions for Large Deflection of Elastic Non-Prismatic Beams" [1] www.facsystems.com/Elastica to obtain new buckling criterion for elastic buckling and ultimate buckling, and the problem will be solved in general in both cases. What we mean elastic buckling is the standard criterion that has been established previously as in Timoshenko and Gere [2] and what is normally derived is the effective length factor. The ultimate buckling criterion is when the elastic buckling load has been exceeded temporally or permanently for some reason or another. For example ultimate buckling can happen due to human error as in crane loads or unpredicted work of nature. In many case depending on the designer the yield is reached then things go back to normal after unloading. But for simple cantilever column depending on the buckling criterion that is picked the load will exceed the yield beyond the safety factors established by the code and it is left to the designer to make these decision after being aware of the forgoing general formulas.

Case I – Prismatic Beam

a) Cantilever Beam

Starting with Eq. 5 in author paper [1] for Fig 1

$$\frac{M(y)}{EI} = -g(y) = \frac{(y_0 - y)P}{EI} = k^2(y_0 - y) \quad \ldots\ldots\ldots\ldots \quad (1)$$

Where $k^2 = \dfrac{P}{EI}$

By substituting Eq.1 in Eq.6 and Eq. 7 ref [1] yields,

$$\dot{x} = \frac{-0.5k^2(y_0 - y)^2 + C1}{\sqrt{1 - \left[-0.5k^2(y_0 - y)^2 + C1\right]^2}} \quad \ldots\ldots\ldots\ldots \quad (2)$$

At $y = 0$ $\quad \dot{x} \Rightarrow -\infty$ or $\dot{y} = 0$ for stiff end condition of a cantilever beam. Thus, the denominator goes to zero. Or

$$C1 = -1 + 0.5k^2 y_0^2 \quad \ldots\ldots\ldots\ldots \quad (3)$$

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Substituting Eq. 3 in Eq. 2, yield;

$$\dot{x} = \frac{-0.5k^2\left[(y_0 - y)^2 - y_0^2\right] - 1}{\sqrt{1 - \left[0.5k^2\left[(y_0 - y)^2 - y_0^2\right] + 1\right]^2}} \qquad (4)$$

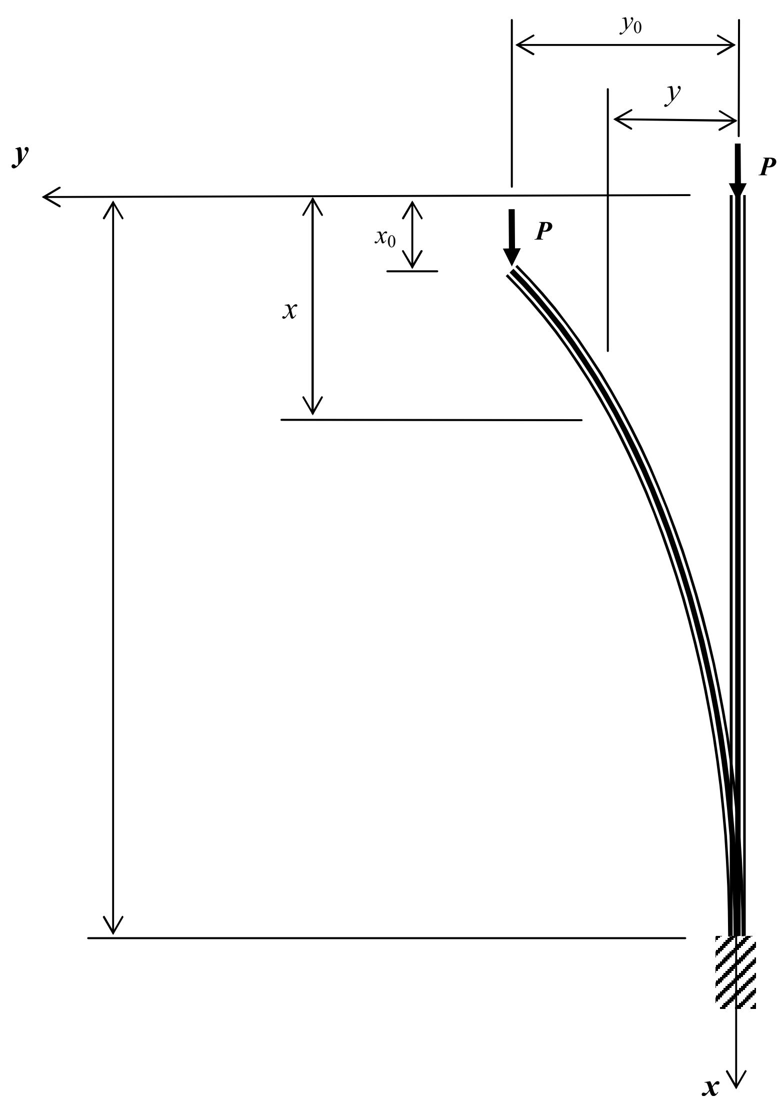

Fig 1 –Cantilever Beam

The length of the beam becomes,

$$L = \int_0^{y_0} \frac{dy}{\sqrt{1 - \left[0.5k^2\left[(y_0 - y)^2 - y_0^2\right] + 1\right]^2}} \quad \ldots\ldots (5)$$

And the deflection can be written as:

$$\int_{x_0}^{L} \dot{x} dy = \int_{y_0}^{0} \frac{-0.5k^2\left[(y_0 - y)^2 - y_0^2\right] - 1}{\sqrt{1 - \left[0.5k^2\left[(y_0 - y)^2 - y_0^2\right] + 1\right]^2}} dy$$

$$L - x_0 = \int_0^{y_0} \frac{0.5k^2\left[(y_0 - y)^2 - y_0^2\right] + 1}{\sqrt{1 - \left[0.5k^2\left[(y_0 - y)^2 - y_0^2\right] + 1\right]^2}} dy$$

$$= \int_0^{y_0} \frac{0.5k^2\left[(y_0 - y)^2 - y_0^2\right]}{\sqrt{1 - \left[0.5k^2\left[(y_0 - y)^2 - y_0^2\right] + 1\right]^2}} dy + \int_0^{y_0} \frac{dy}{\sqrt{1 - \left[0.5k^2\left[(y_0 - y)^2 - y_0^2\right] + 1\right]^2}} \quad \ldots. (6)$$

Substitute Eq. yields,

$$L - x_0 = \int_0^{y_0} \frac{0.5k^2\left[(y_0 - y)^2 - y_0^2\right]}{\sqrt{1 - \left[0.5k^2\left[(y_0 - y)^2 - y_0^2\right] + 1\right]^2}} dy + L$$

Thus:

$$x_0 = -\int_0^{y_0} \frac{0.5k^2\left[(y_0 - y)^2 - y_0^2\right]}{\sqrt{1 - \left[0.5k^2\left[(y_0 - y)^2 - y_0^2\right] + 1\right]^2}} dy \quad \ldots\ldots (7)$$

Let:

$$y_0 - y = y_0 \cos\phi$$
$$-dy = -y_0 \sin\phi \quad \ldots\ldots (8)$$
and let
$$p = 0.5ky_0$$

Substitute in Eq. 4 , Eq. 5 and Eq. 7 yields:

$$\dot{x} = \frac{0.5k^2 y_0^2 \sin^2\phi - 1}{y_0 k \sin\phi \sqrt{1 - p^2 \sin^2\phi}} \quad \ldots\ldots (9)$$

$$\text{at } y = y_0 \text{ or } \phi = \pi/2 \quad \Rightarrow \quad \dot{x} = \cot\alpha_0 = \frac{2p^2 - 1}{2p\sqrt{1-p^2}} \quad \text{and} \quad p = -\sin\left(\frac{\alpha_0}{2}\right) \quad \text{......} \quad (10)$$

$$kL = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - p^2 \sin^2\phi}} = F\left(p, \frac{\pi}{2}\right) \quad \text{......} \quad (11)$$

$$x_0 = \int_0^{\pi/2} \frac{2p^2 \sin^2\phi d\phi}{k\sqrt{1 - p^2 \sin^2\phi}} = \frac{y_0}{p}\left[\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - p^2 \sin^2\phi}} - \int_0^{\pi/2} \sqrt{1 - p^2 \sin^2\phi}\right] = \frac{y_0}{p}\left[F\left(p, \frac{\pi}{2}\right) - E\left(p, \frac{\pi}{2}\right)\right] \quad \text{......} \quad (12)$$

Where the function F is the elliptic integral of the first kind and the function E is the elliptic integral of the second kind.

Solving the equations for various p values we have several buckling conditions summarized in Table 1. and showing the consequences in Fig. 2. Where the load P is the design load per Elastic Buckling following Timoshenko and Gere [2]. The value in Table 1 confirm with the Table 2-4 of Timoshenko and Gere [2].

Table 1 - Buckling Criterion

p	$\sin^{-1}p$	kL	K	$P_{critical} / [\pi^2 EI/(2L)^2]$	y_0 / L	x_0 / L	dx/dy	dy/dx	α_0
0.0000	0.0000	1.5708	2.0000	1.0000	0.0000	0.0000	- ∞	0.0000	0.0000
0.7071	45.0000	1.8541	1.6944	1.3932	0.7628	0.5431	0.0000	- ∞	±90
0.9089	65.3538	2.3210	1.3535	2.1833	0.7832	1.0000	0.8604	0.9269	42.8285
0.9998	89.0000	5.4349	0.5780	11.9714	0.3679	1.6295	28.6363	0.0349	1.9994

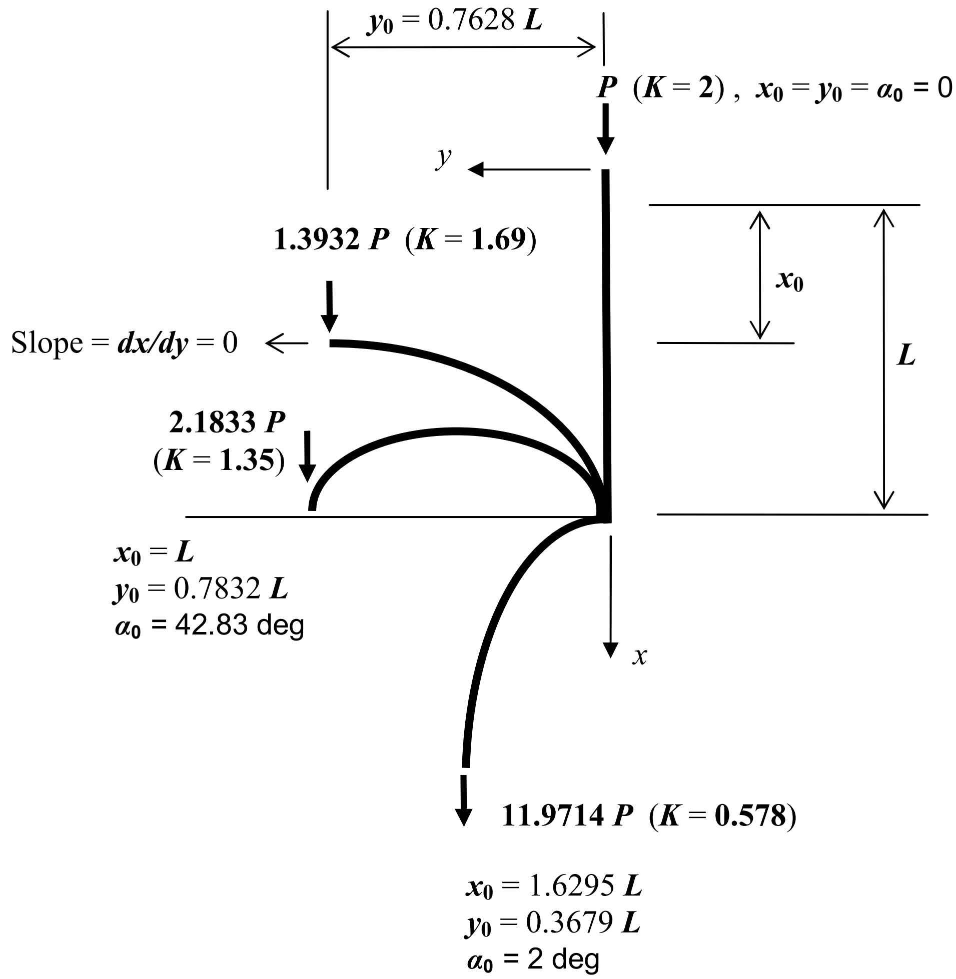

Fig 2 – Buckling Due to Increase in P or reduction of effective length factor K

Discussion over the ± of Eq. 7 of article at www.facsystems.com/Elastica.pdf

Fig. 3 shows the situation

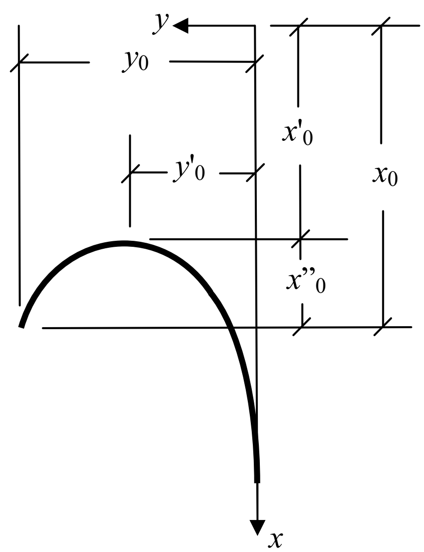

Fig. 3 Change of sign in $y(x)$

$$L - x_0 = L - (x_0' + x_0'') = L - \int_0^{y_0'} \frac{\int g(y) + C1}{\sqrt{1 - \left(\int g(y) + C1\right)^2}} - \int_{y_0'}^{y_0} \frac{-\left[\int g(y) + C1\right]}{\sqrt{1 - \left(\int g(y) + C1\right)^2}}$$

or ………. (13)

$$L - x_0 = L - \int_0^{y_0} \frac{\int g(y) + C1}{\sqrt{1 - \left(\int g(y) + C1\right)^2}}$$

Thus Eq. 12 is correct.

Analysis:

Before solving the general solution for ultimate and linear buckling, we will show two examples that set up the equations. In these examples we show the critical load must be separated from other loads. For example Fig. 4 shows a crane tower where the dead load and moment should be separated from the critical load.

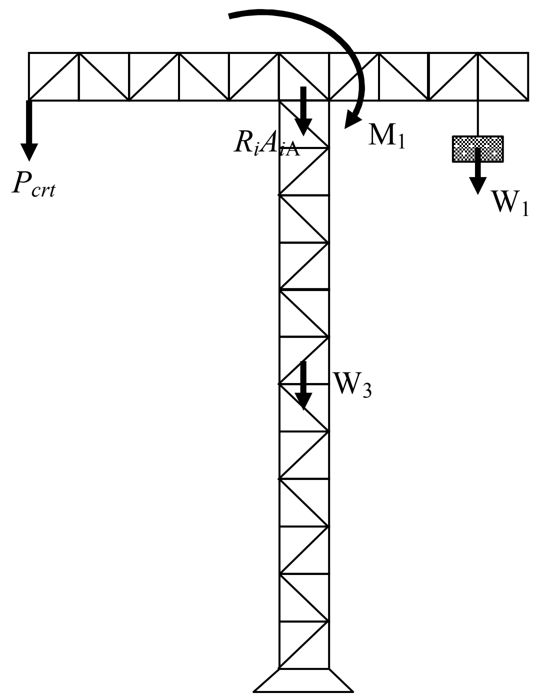

Fig. 4 Crane Loading with *P* critical

A second example in Fig. 5 shows a mechanical bracket with the critical point loads at point E and F. Also, attached is two strings from AB and AC and point D has a hinge. From Fig. 5 we have,

$$P_1 = \frac{2b}{a+b} P_{crt} \quad \text{and} \quad P_2 = \frac{2a}{a+b} P_{crt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(14)}$$

Thus, the buckling on the column is effected by P_1 and P_2 where they are both dependent on P_{crt} . Therefore, we need to set up the buckling equations using different loads attached on the column with a scalar factor multiplied by $P_{crt.}$ We will use t_i as the multiplication factor. In Fig. 5 t_i becomes.

$$\vec{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \dfrac{2b}{a+b} \\ \\ \dfrac{2a}{a+b} \end{bmatrix} \qquad \text{where} \qquad \vec{P} = \vec{t}\,P_{crt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(15)}$$

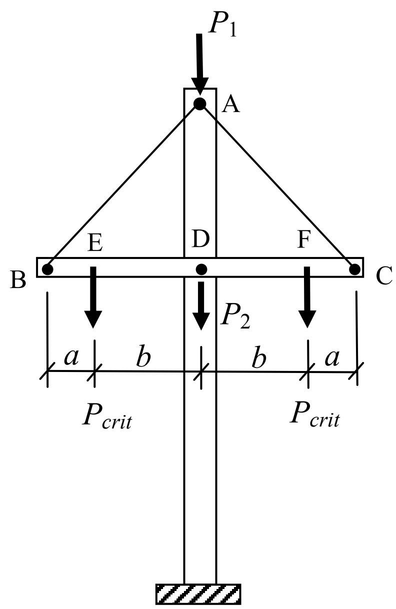

Fig. 5 Mechanical Bracket with two P_{crt}

With these examples we set up the loads at every joint *i* as follows:

P_{Di} and Q_{Di} for dead loads at joint *i*

………………………………………….. (16)

$P_i = t_i P_{crt}$ for live load at joint *i*

Solving Case II of article www.facsystems.com/Elastica.pdf the moments becomes

$$M_i = \sum_{j=0}^{i} \left(P_{crt} t_i + P_{Di}\right)\left(y_j - y\right) + Q_{Dj} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(17)}$$

$$g_i(y) = -\frac{1}{EI_i}\left[\sum_{j=0}^{i}\left(P_{crt}t_i + P_{Di}\right)\left(y_j - y\right) + Q_{Dj}\right]$$

or

$$g_i(y) = -\frac{1}{EI_i}\left[R_i\left(y_i - y\right) + A_i R_i\right]$$

where …………………………………………… (18)

$$R_i = \sum_{j=0}^{i}\left(P_{crt}t_j + P_{Dj}\right)$$

and

$$A_i = \frac{1}{R_i}\left[\sum_{j=0}^{i}\left(P_{crt}t_i + P_{Di}\right)\left(y_j - y_i\right) + Q_{Dj}\right]$$

Integrating $g_i(y)$ yields:

$$\int g_i(y) = \frac{1}{EI_i}\left[0.5R_i\left(y_i - y\right)^2 + A_i R_i\left(y_i - y\right) + \left(C1_i - 1\right)EI_i\right] + 1$$

or

$$\int g_i(y) = \frac{0.5R_i}{EI_i}\left[\left(y_i - y + A_i\right)^2 - B_i^2\right] + 1 \qquad \text{…………………………. (19)}$$

where

$$B_i = \sqrt{A_i^2 - \frac{2\left(C1_i - 1\right)EI_i}{R_i}} = A_i\sqrt{1 - \frac{2\left(C1_i - 1\right)EI_i}{A_i^2 R_i}}$$

And the equations become:

$$L_i = \int_{y_{i+1}}^{y_i} \frac{dy}{\sqrt{1 - \left[\frac{0.5R_i}{EI_i}\left[\left(y_i - y - A_i\right)^2 - B_i^2\right] + 1\right]^2}} \qquad \text{……………………………………….. (20)}$$

$$\dot{x}_i = -\frac{\frac{0.5R_i}{EI_i}\left[\left(y_i - y - A_i\right)^2 - B_i^2\right] + 1}{\sqrt{1 - \left[\frac{0.5R_i}{EI_i}\left[\left(y_i - y - A_i\right)^2 - B_i^2\right] + 1\right]^2}} \qquad \text{………………………………………… (21)}$$

$$\int_{x_i}^{x_{i+1}} \dot{x}_i(y) = x_{i+1} - x_i = \int_{y_i}^{y_{i+1}} \frac{\frac{0.5R_i}{EI_i}\left[(y_i - y - A_i)^2 - B_i^2\right] + 1}{\sqrt{1 - \left[\frac{0.5R_i}{EI_i}\left[(y_i - y - A_i)^2 - B_i^2\right] + 1\right]^2}} dy$$

or

$$x_{i+1} - x_i = \int_{y_i}^{y_{i+1}} \frac{\frac{0.5R_i}{EI_i}\left[(y_i - y - A_i)^2 - B_i^2\right]}{\sqrt{1 - \left[\frac{0.5R_i}{EI_i}\left[(y_i - y - A_i)^2 - B_i^2\right] + 1\right]^2}} dy + L_i \qquad \text{.................. (22)}$$

Now:

$$(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + ... + (x_1 - x_0) = \sum_{i=0}^{n-1} \int_{y_i}^{y_{i+1}} \frac{\frac{0.5R_i}{EI_i}\left[(y_i - y - A_i)^2 - B_i^2\right]}{\sqrt{1 - \left[\frac{0.5R_i}{EI_i}\left[(y_i - y - A_i)^2 - B_i^2\right] + 1\right]^2}} dy + L$$

or for $x_n = L$ we have

$$x_0 = -\sum_{i=0}^{n-1} \int_{y_i}^{y_{i+1}} \frac{\frac{0.5R_i}{EI_i}\left[(y_i - y - A_i)^2 - B_i^2\right]}{\sqrt{1 - \left[\frac{0.5R_i}{EI_i}\left[(y_i - y - A_i)^2 - B_i^2\right] + 1\right]^2}} dy$$

.... (23)

The change in sign in Eq. 7 of article www.facsystems.com/Elastica.pdf is included as discussed before in Eq. 13. The coefficient $C1_{n-1}$ is found at $y = y_n = 0$, $\dot{x}_n = \infty$, thus

$$\frac{0.5R_{n-1}}{EI_{n-1}}\left[(y_{n-1} - 0 - A_{n-1})^2 - B_{n-1}^2\right] = 0$$

or

$$B_{n-1}^2 = (y_{n-1} - A_{n-1})^2 \qquad \text{... (24)}$$

or from Eq.19

$$C1_{n-1} = \frac{R_{n-1}}{2EI_{n-1}}\left[A_{n-1}^2 - (y_{n-1} - A_{n-1})^2\right] + 1$$

Now let

$$y_i - y + A_i = B_i \cos\phi$$
$$dy = B_i \sin\phi \, d\phi$$
$$\phi 2_i = \cos^{-1}\left[\frac{A_i}{B_i}\right]$$
$$\phi 1_{i+1} = \cos^{-1}\left[\frac{y_i - y_{i+1} + A_i}{B_i}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$
$$k_i^2 = \frac{R_i}{EI_i}$$
and
$$p_i = 0.5 k_i B_i$$

In Eq. 20, Eq. 21 and Eq. 23 yields:

$$L = \sum_{i=0}^{n-1} \frac{1}{k_i} \int_{\phi 1_{i+1}}^{\phi 2_i} \frac{d\phi}{\sqrt{1 - p_i^2 \sin^2\phi}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

And Eq. 26 becomes an elliptical integral.

Similarly:

$$\dot{x}_0 = \frac{2 p_0^2 \sin^2\phi_0 - 1}{2 p_0 \sin\phi_0 \sqrt{1 - p_0^2 \sin^2\phi_0}}$$
where
$$p_0 = 0.5 k_0 B_0$$
$$R_0 = P_{crt} t_0 + P_{D0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$
$$A_0 = \frac{Q_{D0}}{R_0} \quad \text{and} \quad B_0 = \sqrt{A_0^2 + \frac{2(1 - C1_0) EI_0}{R_0}}$$
$$\phi_0 = \cos^{-1}\left(\frac{A_0}{B_0}\right)$$

Thus:

$$\dot{x}_0 = 0 \quad \text{or} \quad \frac{dy}{dx} = \infty \Big|_{@\, x = x_0 \text{ and } y = y_0} = \quad \text{at}$$
$$p_0 \sin\phi_0 = \frac{1}{\sqrt{2}}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

This becomes the ultimate buckling for a zero slope or $\frac{dy}{dx} = \infty$ at the tip of the column.

The procedure is to pick $P = P_{crt}$ and solve for y_0 , y_1 , …….., y_{n-1} from Eq. 20 through L_i (similar to Eq. 27 or appendix A in article www.facsystems.com/Elastica.pdf) and check if Eq. 28 is satisfied else update P_{crt} , and P_{crt} for the first condition of ultimate buckling can be obtained. Now for the second condition of the ultimate buckling namely P_{crt} for $x_0 = L$ can be found from writing Eq. 23 using Eq. 25 yields;

$$\begin{aligned} x_0 = L &= \sum_{i=0}^{n-1} \frac{2}{k_i} \int_{\phi 1_{i+1}}^{\phi 2_i} \frac{p_i^2 \sin^2 \phi \, d\phi}{\sqrt{1 - p_i^2 \sin^2 \phi}} \\ &= \sum_{i=0}^{n-1} \frac{2}{k_i} \left[\int_{\phi 1_{i+1}}^{\phi 2_i} \frac{d\phi}{\sqrt{1 - p_i^2 \sin^2 \phi}} - \int_{\phi 1_{i+1}}^{\phi 2_i} \sqrt{1 - p_i^2 \sin^2 \phi} \, d\phi \right] \end{aligned} \qquad \text{……………………… (29)}$$

And Eq. 29 becomes elliptical integrals. The procedure is to pick $P = P_{crt}$ and solve for y_0 , y_1 , …….., y_{n-1} from Eq. 20 through L_i (similar to Eq. 27 or appendix A in article www.facsystems.com/Elastica.pdf) and check if Eq. 29 is satisfied else update P_{crt} , and P_{crt} for the second condition of ultimate buckling can be obtained.

In these analysis the coefficient $C1_i$ can be found from B_i by forcing the condition on equal slope at the joints. Thus from Eq. 21:

$$\frac{0.5R_{i-1}}{EI_{i-1}} \left[(y_{i-1} - y_i + A_{i-1})^2 - B_{i-1}^2 \right] + 1 = \frac{0.5R_i}{EI_i} \left[A_i^2 - B_i^2 \right] + 1$$

or

$$\begin{aligned} B_{i-1}^2 &= -\frac{EI_{i-1} R_i}{EI_i R_{i-1}} \left[A_i^2 - B_i^2 \right] + (y_{i-1} - y_i + A_{i-1})^2 \\ &= \frac{k_i^2}{k_{i-1}^2} \left[A_i^2 - B_i^2 \right] + (y_{i-1} - y_i + A_{i-1})^2 \end{aligned} \qquad \text{………………………….. (30)}$$

where

$i = 0, 1, \ldots\ldots\ldots (n-1)$

and from Eq. 24

$$B_{n-1}^2 = (y_{n-1} - A_{n-1})^2$$

$$C1_{n-1} = \frac{R_{n-1}}{2EI_{n-1}} \left[A_{n-1}^2 - (y_{n-1} - A_{n-1})^2 \right] + 1$$

For the condition where it is a continuous moment of inertia for a non-prismatic column then set $I_{i-1} = I_i$ in Eq. 30 for the moment of inertia at the joints.

Linear Buckling per Timoshenko:

This is the case where P_{crt} is to be found for $y_i = 0$. We will have to start with an example from Timoshenko p114 to set up the solution. From Timoshenko Eq 114 and Fig 6, we have

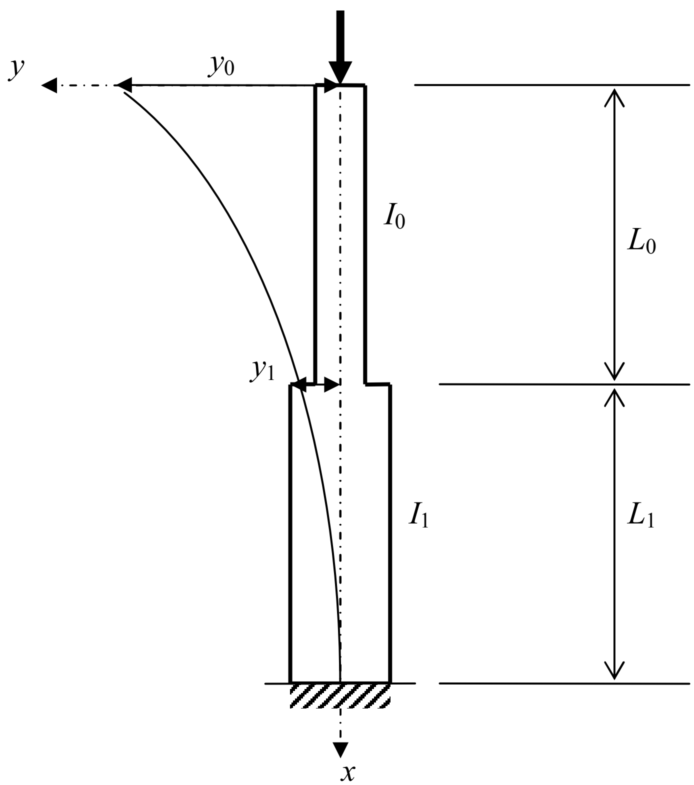

Fig 6 Timoshenko Example

$$y_1 = y_0\left(1-\cos k_1 L_1\right)$$

or

$$\frac{y_0 - y_1}{y_0} = \cos k_1 L_1 \qquad \text{(31)}$$

where

$$k_0^2 = \frac{P}{EI_0} \quad \text{and} \quad k_1^2 = \frac{P}{EI_1}$$

Using Eq. 26 we have:

$$L = \frac{1}{k_0}\int_{\phi 1_1}^{\phi 2_0} \frac{d\phi}{\sqrt{1-p_0^2 \sin^2\phi}} + \frac{1}{k_1}\int_{\phi 1_2}^{\phi 2_1} \frac{d\phi}{\sqrt{1-p_1^2 \sin^2\phi}} \qquad \text{(32)}$$

We have from Eq. 18 and Eq. 19

$$P_{Dj} = Q_{Dj} = 0, \quad t_0 = 1, \quad t_1 = 0, \quad R_0 = P, \quad R_1 = P, \quad A_0 = 0, \quad A_1 = y_0 - y_1,$$

$$B_0^2 = \frac{2(1-C1_0)EI_0}{P}, \qquad B_1^2 = \left(y_0 - y_1\right)^2 + \frac{2(1-C1_1)EI_1}{P},$$

And from Eq. 24 and Eq. 30

$$B_1 = y_1 + A_1 = y_0$$

$$B_0^2 = -\frac{I_0}{I_1}\left(A_1^2 - B_1^2\right) + \left(y_0 - y_1 + A_0\right)^2 = -\frac{I_0}{I_1}\left[\left(y_0 - y_1\right)^2 - y_0^2\right] + \left(y_0 - y_1\right)^2$$

As $y_0 = y_1$ approch zero $\Rightarrow B_0 = B_1 \to 0$ or $k_0 B_0 = k_1 B_1 \to 0$ or $p_0 = p_1 \to 0$

Also from Eq. :

$$\phi 2_0 = \cos^{-1}\left[\frac{A_0}{B_0}\right] = \cos^{-1}(0) = \frac{\pi}{2}$$

$$\phi 1_1 = \cos^{-1}\left[\frac{y_0 - y_1 + A_0}{B_0}\right] = \cos^{-1}\left(\frac{y_0 - y_1}{\sqrt{-\frac{I_0}{I_1}\left[(y_0 - y_1)^2 - y_0^2\right] + (y_0 - y_1)^2}}\right)$$

$$= \cos^{-1}\left[\frac{\frac{y_0 - y_1}{y_0}}{\sqrt{-\frac{I_0}{I_1}\left[\left(\frac{y_0 - y_1}{y_0}\right)^2 - 1\right] + \left(\frac{y_0 - y_1}{y_0}\right)^2}}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

$$\phi 2_1 = \cos^{-1}\left[\frac{A_1}{B_1}\right] = \cos^{-1}\left(\frac{y_0 - y_1}{y_0}\right)$$

$$\phi 1_2 = \cos^{-1}\left[\frac{y_1 - y_2 + A_1}{B_1}\right] = \cos^{-1}\left[\frac{y_1 - 0 - (y_0 - y_1)}{y_0}\right] = \cos^{-1}(1) = 0$$

We see in the first integral on the right hand side of Eq. 32 shows:

$$\cos^{-1}\left[\frac{\frac{y_0 - y_1}{y_0}}{\sqrt{-\frac{I_0}{I_1}\left[\left(\frac{y_0 - y_1}{y_0}\right)^2 - 1\right] + \left(\frac{y_0 - y_1}{y_0}\right)^2}}\right]_{\text{as } y_0 = y_1 \to 0} \to \frac{\pi}{2} \le \phi \le \frac{\pi}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

Or $\phi \to 0$ in the first integral as $y_0 = y_1 \to 0$ thus $p_0^2 \sin^2 \phi \to 0$ in the denominator.

Similarly we see in the second integral on the right hand side of Eq. 32 shows:

$$0 \le \phi \le \cos^{-1}\left(\frac{y_0 - y_1}{y_0}\right)_{\text{as } y_0 = y_1 \to 0} \to 0$$

Or $\phi \to 0$ in the second integral as $y_0 = y_1 \to 0$ thus $p_1^2 \sin^2 \phi \to 0$ in the denominator.

Thus Eq. 32 becomes:

$$
\begin{aligned}
L &= \frac{1}{k_0}\int_{\phi 1_1}^{\phi 2_0} d\phi + \frac{1}{k_1}\int_{\phi 1_2}^{\phi 2_1} d\phi \\
&= \frac{\phi 2_0 - \phi 1_1}{k_0} + \frac{\phi 2_1 - \phi 1_2}{k_1}
\end{aligned}
\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)
$$

Substituting Eq. 31 in Eq. 33 and then substituting in Eq. 35 yields;

$$
\begin{aligned}
L &= \frac{1}{k_0}\left\{\frac{\pi}{2} - \cos^{-1}\left[\frac{\cos k_1 L_1}{\sqrt{-\frac{I_0}{I_1}\left[\cos^2 k_1 L_1 - 1\right] + \cos^2 k_1 L_1}}\right]\right\} + \frac{1}{k_1}\left(k_1 L_1 - 0\right) \\
&= \frac{1}{k_0}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{\frac{I_0}{I_1}}\sin k_1 L_1}{\cos k_1 L_1}\right)\right] + L_1
\end{aligned}
$$

or

$$
k_0 L_0 = \frac{\pi}{2} - \tan^{-1}\left(\sqrt{\frac{I_0}{I_1}}\tan k_1 L_1\right)
$$

which can be written as :

$$
\tan k_1 L_1 \tan k_0 L_0 = \sqrt{\frac{I_1}{I_0}} = \frac{k_0}{k_1} \quad \ldots\ldots\ldots\ldots\ldots (36)
$$

Equation 36 matches Eq. (b) pp 114 of Timoshenko.

<u>General Solution for Linear Buckling:</u>

To obtain Timoshenko linear buckling we need to set $Q_{Dj} = 0$ in Eq. 18 as explained in chapter 1 of Timoshenko book the moments and loads in the x direction can be isolated from the vertical load. Thus:

$$
A_i = \frac{1}{R_i}\left[\sum_{j=0}^{i}\left(P_{crt} t_i + P_{Di}\right)\left(y_j - y_i\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)
$$

We first note as y_i approach zero:

$B_{n-1} = A_{n-1}$ in Eq. 24

$B_{n-2} = A_{n-2}$ in Eq. 30

.

.

$B_i = A_i$ from Eq. 30 ………………………………………………………………… (38)

.

$B_1 = A_1$

$B_0 = A_0 = 0, \quad \dfrac{A_0}{B_0} = 0$

Thus from Eq. 25

$$\phi 1_{i+1} = \cos^{-1}\left[\frac{y_i - y_{i+1} + A_i}{B_i}\right]_{\text{as } y_i \to 0} = \cos^{-1}(1) \le \phi \le \phi 2_i = \cos^{-1}\left[\frac{A_i}{B_i}\right] = \cos^{-1}(1)$$

Thus $\phi \to 0$ as $y_i \to 0$ Also $B_i = A_i = 0$ as $y_i \to 0$ in Eq. 18

Thus $p_i = 0.5 k_i B_i \to 0$ in Eq. 25 . Therefore $p_i^2 \sin^2\phi \to 0$ in the denominator of Eq. 26.

And Eq. 26 becomes:

$$\begin{aligned} L &= \sum_{i=0}^{n-1} \frac{1}{k_i} \int_{\phi 1_{i+1}}^{\phi 2_i} d\phi \\ &= \sum_{i=0}^{n-1} \frac{\phi 2_i - \phi 1_{i+1}}{k_i} \end{aligned} \quad \text{……………………………………………………………………} \quad (39)$$

Or from Eq. 25 yields;

$$L = \sum_{i=0}^{n-1} \frac{1}{k_i}\left\{\cos^{-1}\left(\frac{A_i}{B_i}\right) - \cos^{-1}\left(\frac{y_i - y_{i+1} + A_i}{B_i}\right)\right\} \quad \text{………………………………………} \quad (40)$$

Now we need to relate A_i , B_i and y_i to $k_i L_i$. To start we look at every segment L_i and note the curvature can be approximated to the differential equation:

$$EI_i \frac{d^2 y}{dx^2} = \sum_{j=0}^{i} \left(Pt_j + P_{Dj}\right)\left(y_j - y\right) = R_i A_i - R_i y$$

or

$$EI_i \frac{d^2 y}{dx^2} + R_i y = R_i A_i \qquad \text{……………………………………….. (41)}$$

or

$$\frac{d^2 y}{dx^2} + k_i^2 y = \frac{R_i A_i}{EI_i}$$

First we translate the axis with

$$y = y' + y_{i+1} \quad \text{and} \quad x = -x' + x_{i+1}$$

so …………………………. (42)

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dy'}\frac{dy'}{dx'}\frac{dx'}{dx}\right] = -\frac{d}{dx}\left[\frac{dy'}{dx'}\right] = -\frac{dx'}{dx}\frac{d^2 y'}{dx'^2} = \frac{d^2 y'}{dx'^2}$$

Substituting in Eq. 41 yields;

$$\frac{d^2 y'}{dx'^2} + k_i^2 (y' + y_{i+1}) = \frac{R_i A_i}{EI_i}$$

or ………………………………………………..…………….. (43)

$$\frac{d^2 y'}{dx'^2} + k_i^2 y' = \frac{R_i (A_i - y_{i+1})}{EI_i}$$

Next we rotate the axis by α_{i+1} where α_{i+1} is the angle at the bottom of the segment L_i at joint i+1. This makes the segment fixed at joint i+1. Thus:

$$\begin{aligned} y'' &= y' \cos\alpha_{i+1} - x' \sin\alpha_{i+1} && \text{and} && y' = y'' \cos\alpha_{i+1} + x'' \sin\alpha_{i+1} \\ x'' &= y' \sin\alpha_{i+1} + x' \cos\alpha_{i+1} && \text{and} && x' = -y'' \sin\alpha_{i+1} + x'' \cos\alpha_{i+1} \end{aligned} \qquad \text{……………….. (44)}$$

Instead of substituting Eq. 44 in Eq. 43 we look at a free body diagram, Fig 7, and resolve force R_i with the rotational angle α_{i+1} at joint i into the fixed base segment L_i along with the carried moment $R_i A_i$.

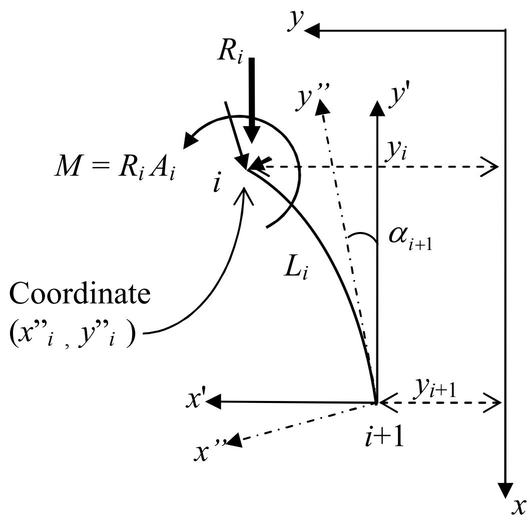

Fig 7 – Free Body Diagram

The differential equation becomes:

$$EI_i \frac{d^2 y''}{dx''^2} = R_i A_i + (y''_i - y'')R_i \cos\alpha_{i+1} + (x''_i - x'')R_i \sin\alpha_{i+1} \quad \text{.......................... (45)}$$

When $\alpha_{i+1} = 0 \quad \Rightarrow \quad y''_i = y'_i = y_i - y_{i+1}$ and $y'' = y'$ and Eq. 45 becomes Eq. 43.

Rewite Eq. 45 to

$$EI_i \frac{d^2 y''}{dx''^2} + y'' R_i \cos\alpha_{i+1} = R_i A_i + y''_i\, R_i \cos\alpha_{i+1} + x''_i\, R_i \sin\alpha_{i+1} - x'' R_i \sin\alpha_{i+1}$$

or

$$\frac{d^2 y''}{dx''^2} + y'' k_i^2 \cos\alpha_{i+1} = k_i^2\left(A_i + y''_i \cos\alpha_{i+1} + x''_i \sin\alpha_{i+1}\right) - x'' k_i^2 \sin\alpha_{i+1}$$

from Eq. 44 we have …….. (46)

$$\frac{d^2 y''}{dx''^2} + y'' k_i^2 \cos\alpha_{i+1} = k_i^2\left(A_i + y'_i\right) - x'' k_i^2 \sin\alpha_{i+1}$$

from Eq. 42 we have

$$\frac{d^2 y''}{dx''^2} + y'' k_i^2 \cos\alpha_{i+1} = k_i^2\left(A_i + y_i - y_{i+1}\right) - x'' k_i^2 \sin\alpha_{i+1}$$

The solution to Eq. 46 is

$$y'' = A + Bx'' + C\cos k_i x''\sqrt{\cos\alpha_{i+1}} + D\sin k_i x''\sqrt{\cos\alpha_{i+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (47)$$

Substituting Eq. 47 in Eq. 46 yields;

$$Ak_i^2\cos\alpha_{i+1} = k_i^2\left(A_i + y_i - y_{i+1}\right)$$

or

$$A = \frac{A_i + y_i - y_{i+1}}{\cos\alpha_{i+1}}$$

and

$$Bk_i^2\cos\alpha_{i+1} = -k_i^2\sin\alpha_{i+1}$$

or

$$B = -\tan\alpha_{i+1}$$

Thus Eq. 47 becomes:

$$y'' = \frac{A_i + y_i - y_{i+1}}{\cos\alpha_{i+1}} - \tan\alpha_{i+1}x'' + C\cos k_i x''\sqrt{\cos\alpha_{i+1}} + D\sin k_i x''\sqrt{\cos\alpha_{i+1}} \quad \ldots\ldots\ldots\ldots\ (48)$$

Enforce the end condition:

$@\, x'' = 0$ and $y'' = 0$ Eq. 48 must be satisfied or

$$C = -\frac{A_i + y_i - y_{i+1}}{\cos\alpha_{i+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (49)$$

And Eq. 49 becomes:

$$y'' = \frac{A_i + y_i - y_{i+1}}{\cos\alpha_{i+1}}\left(1 - \cos k_i x''\sqrt{\cos\alpha_{i+1}}\right) + D\sin k_i x''\sqrt{\cos\alpha_{i+1}} - \tan\alpha_{i+1}x'' \quad \ldots\ldots\ldots\ldots\ (50)$$

Differentiating Eq. 50 yields;

$$\frac{dy''}{dx''} = \frac{A_i + y_i - y_{i+1}}{\cos\alpha_{i+1}} k_i\sqrt{\cos\alpha_{i+1}}\sin k_i x''\sqrt{\cos\alpha_{i+1}} + Dk_i\sqrt{\cos\alpha_{i+1}}\cos k_i x''\sqrt{\cos\alpha_{i+1}} - \tan\alpha_{i+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (51)$$

Enforce another end condition:

$$@\, x''=0 \text{ and } y''=0 \quad \frac{dy''}{dx''}=0$$

$$D=\frac{\tan\alpha_{i+1}}{k_i\sqrt{\cos\alpha_{i+1}}} \qquad \text{.......... (52)}$$

And Eq. 51 becomes:

$$\frac{dy''}{dx''}=\frac{A_i+y_i-y_{i+1}}{\cos\alpha_{i+1}}k_i\sqrt{\cos\alpha_{i+1}}\sin k_i x''\sqrt{\cos\alpha_{i+1}}+\tan\alpha_{i+1}\left(\cos k_i x''\sqrt{\cos\alpha_{i+1}}-1\right) \qquad \text{.......... (53)}$$

And the solution to the differential Eq. 46 becomes:

$$y''=\frac{A_i+y_i-y_{i+1}}{\cos\alpha_{i+1}}\left(1-\cos k_i x''\sqrt{\cos\alpha_{i+1}}\right)+\frac{\tan\alpha_{i+1}}{k_i\sqrt{\cos\alpha_{i+1}}}\sin k_i x''\sqrt{\cos\alpha_{i+1}}-\tan\alpha_{i+1}x'' \qquad \text{.......... (54)}$$

Substituting $y''=y''_i$ and $x''=x''_i=L_i$ in Eq. 54 and rearranging yield;

$$\cos\alpha_{i+1}y''_i+\sin\alpha_{i+1}x''=\left(A_i+y_i-y_{i+1}\right)\left(1-\cos k_i L_i\sqrt{\cos\alpha_{i+1}}\right)+\frac{\sin\alpha_{i+1}}{k_i\sqrt{\cos\alpha_{i+1}}}\sin k_i L_i\sqrt{\cos\alpha_{i+1}}$$

or from Eq. 42 and Eq. 44 we have:

$$y_i-y_{i+1}=A_i+y_i-y_{i+1}-\left(A_i+y_i-y_{i+1}\right)\cos k_i L_i\sqrt{\cos\alpha_{i+1}}+\frac{\sin\alpha_{i+1}}{k_i\sqrt{\cos\alpha_{i+1}}}\sin k_i L_i\sqrt{\cos\alpha_{i+1}} \qquad \text{.......... (55)}$$

Cancelling terms in Eq. 55 yields:

$$\frac{A_i+y_i-y_{i+1}}{A_i}=\frac{1}{\cos k_i L_i\sqrt{\cos\alpha_{i+1}}}+\frac{\sin\alpha_{i+1}}{A_i k_i\sqrt{\cos\alpha_{i+1}}}\tan k_i L_i\sqrt{\cos\alpha_{i+1}} \qquad \text{.......... (56)}$$

Finding α_{i+1} is as follows:
Rearranging Eq. 53 yields;

$$\frac{dy''}{dx''}\cos\alpha_{i+1}+\sin\alpha_{i+1}=\left(A_i+y_i-y_{i+1}\right)k_i\sqrt{\cos\alpha_{i+1}}\sin k_i x''\sqrt{\cos\alpha_{i+1}}+\sin\alpha_{i+1}\cos k_i x''\sqrt{\cos\alpha_{i+1}} \qquad \text{.......... (57)}$$

Enforing the condition @ $x'' = L_i \rightarrow \frac{dy''}{dx''} = \tan(\alpha_i - \alpha_{i+1})$ in Eq. 57 yields;

$$\tan(\alpha_i - \alpha_{i+1})\cos\alpha_{i+1} + \sin\alpha_{i+1} = (A_i + y_i - y_{i+1})k_i\sqrt{\cos\alpha_{i+1}}\sin k_i L_i\sqrt{\cos\alpha_{i+1}} + \sin\alpha_{i+1}\cos k_i L_i\sqrt{\cos\alpha_{i+1}} \quad \text{……………………………………… (58)}$$

Now as $y_i \rightarrow 0 \quad \cos\alpha_{i+1} \rightarrow 1, \ \sin\alpha_{i+1} \rightarrow \alpha_{i+1}$ and $\tan(\alpha_i - \alpha_{i+1}) \rightarrow \alpha_i - \alpha_{i+1}$
Substituting in Eq. 56 and 58 yields;

$$\frac{A_i + y_i - y_{i+1}}{A_i} = \frac{1}{\cos k_i L_i} + \frac{\alpha_{i+1}}{A_i k_i}\tan k_i L_i \quad \text{………………………………………….. (59)}$$

$$\alpha_i - \alpha_{i+1} + \alpha_{i+1} = (A_i + y_i - y_{i+1})k_i \sin k_i L_i + \alpha_{i+1}\cos k_i L_i$$

or ……………………………………… (60)

$$\alpha_i - \alpha_{i+1}\cos k_i L_i = (A_i + y_i - y_{i+1})k_i \sin k_i L_i$$

Eq. 60 is a finite difference equation and has a solution equal to:

$$\alpha_i = (A_i + y_i - y_{i+1})k_i \sin k_i L_i \left[\frac{1 - (\cos k_i L_i)^{n-i}}{1 - \cos k_i L_i}\right] \quad \text{………………………………….. (61)}$$

So that at $i = n$, $\alpha_i = 0$. Substituting Eq. 61 in Eq. 59 yiels;

$$\frac{A_i + y_i - y_{i+1}}{A_i} = \frac{1}{\cos k_i L_i} + \frac{(A_i + y_i - y_{i+1})}{A_i}\sin k_i L_i \left[\frac{1 - (\cos k_i L_i)^{n-i-1}}{1 - \cos k_i L_i}\right]\tan k_i L_i$$

or ………. (62)

$$\psi_i = \frac{A_i + y_i - y_{i+1}}{A_i} = \frac{1}{\cos k_i L_i}\left[\frac{1}{1 - \sin k_i L_i \tan k_i L_i \left[\frac{1 - (\cos k_i L_i)^{n-i-1}}{1 - \cos k_i L_i}\right]}\right]$$

Substituting Eq. 62 in Eq. 40 yields;

$$L = \sum_{i=0}^{n-1}\frac{1}{k_i}\left\{\cos^{-1}\left(\frac{A_i}{B_i}\right) - \cos^{-1}\left(\psi_i \frac{A_i}{B_i}\right)\right\}_{\text{where } A_i \neq 0} \quad \text{……………………………………… (63)}$$

Now for the first term $A_0 = 0$ and

$$\phi 1_1 = \cos^{-1}\left(\frac{y_0 - y_1}{B_0}\right)$$

and from Eq. 30

$$B_0 = \sqrt{-\frac{EI_0}{EI_1}\frac{R_1}{R_0}\left(A_1^2 - B_1^2\right) + \left(y_0 - y_1\right)^2}$$

or

$$\phi 1_1 = \cos^{-1}\left(\frac{1}{\sqrt{1 - \frac{EI_0}{EI_1}\frac{R_1}{R_0}\frac{A_1^2 - B_1^2}{\left(y_0 - y_1\right)^2}}}\right)$$

Since from Eq. 37 $A_1 = \left(y_0 - y_1\right)$ then

$$\phi 1_1 = \cos^{-1}\left(\frac{1}{\sqrt{1 - \frac{EI_0}{EI_1}\frac{R_1}{R_0}\left(1 - \frac{B_1^2}{A_1^2}\right)}}\right) \qquad \text{.. (64)}$$

Thus:

$$L = \frac{1}{k_0}\left(\frac{\pi}{2} - \cos^{-1}\left[\frac{1}{\sqrt{1 - \frac{EI_0}{EI_1}\frac{R_1}{R_0}\left(1 - \frac{B_1^2}{A_1^2}\right)}}\right]\right) + \sum_{i=1}^{n-1}\frac{1}{k_i}\left\{\cos^{-1}\left(\frac{A_i}{B_i}\right) - \cos^{-1}\left(\psi_i \frac{A_i}{B_i}\right)\right\} \quad \text{..... (65)}$$

From Eq. 30 we have:

$$\begin{aligned}\frac{B_{i-1}}{A_{i-1}} &= \sqrt{\frac{k_i^2}{k_{i-1}^2}\frac{A_i^2 - B_i^2}{A_{i-1}^2} + \frac{\left(y_{i-1} - y_i + A_{i-1}\right)^2}{A_{i-1}^2}} \\ &= \sqrt{\frac{k_i^2}{k_{i-1}^2}\frac{A_i^2 - B_i^2}{A_i^2}\frac{A_i^2}{A_{i-1}^2} + \psi_{i-1}^2} \qquad \text{.. (66)}\end{aligned}$$

Now from Eq. 37 we have:

$$
\begin{aligned}
A_i &= \frac{1}{R_i}\left[\sum_{j=0}^{i}\left(P_{crt}t_i + P_{Di}\right)\left(y_j - y_i\right)\right] \\
&= \frac{1}{R_i}\left[\sum_{j=0}^{i-1}\left(P_{crt}t_i + P_{Di}\right)\left(y_j - y_{i-1}\right)\right] + \frac{1}{R_i}\left[\sum_{j=0}^{i-1}\left(P_{crt}t_i + P_{Di}\right)\left(y_{i-1} - y_i\right)\right] \\
&= \frac{R_{i-1}}{R_i}A_{i-1} + \frac{R_{i-1}}{R_i}\left(y_{i-1} - y_i\right)
\end{aligned}
$$

or ………………… (67)

$$
A_i = \frac{R_{i-1}}{R_i}A_{i-1}\frac{\left(A_{i-1} + y_{i-1} - y_i\right)}{A_{i-1}}
$$

from Eq. 62 we have:

$$
A_i = \frac{R_{i-1}}{R_i}A_{i-1}\psi_{i-1}
$$

Substituting Eq. 67 in 66 yields:

$$
\begin{aligned}
\frac{B_{i-1}}{A_{i-1}} &= \psi_{i-1}\sqrt{\frac{k_i^2}{k_{i-1}^2}\frac{R_{i-1}^2}{R_i^2}\left(1 - \frac{B_i^2}{A_i^2}\right) + 1} \\
&= \psi_{i-1}\sqrt{\frac{R_{i-1}}{R_i}\frac{EI_{i-1}}{EI_i}\left(1 - \frac{B_i^2}{A_i^2}\right) + 1} \qquad \text{………..………………………………………… (68)}
\end{aligned}
$$

$$
\frac{A_{i-1}}{B_{i-1}} = \frac{1}{\psi_{i-1}\sqrt{\frac{R_{i-1}}{R_i}\frac{EI_{i-1}}{EI_i}\left(1 - \frac{B_i^2}{A_i^2}\right) + 1}}
$$

For the end condition from Eq. 24 for $y_{n-1} \to 0$ then $B_{n-1} = A_{n-1}$ or $\frac{A_{n-1}}{B_{n-1}} = 1$, and all $\frac{A_i}{B_i}$ can be found consecutively from Eq. 68 and substituted in Eq. 65 and the solution is found by solving for P_{crt}.

Other End Conditions:

<u>End Condition #1</u>: This condition has a pin at bottom end and rotational fixed at the top end but free to translate at the top as in Fig. 8.

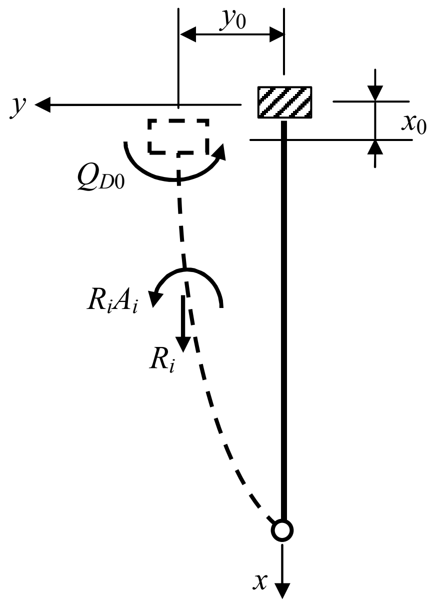

Fig. 8 Pin – Rotational Fixed Column

Thus the moment Q_{D0} at the tip of the column makes $\left.\frac{dy}{dx}=0\right|_{@\,y=x=0}\quad 0\rightarrow\left.\frac{dx}{dy}=\infty\right|_{@\,y=x=0}$

Now at $y=y_n$ and $x=L$ $g_n(y_n)=0$ or Eq. 18 becomes:

$$\sum_{j=0}^{n-1}\left[\left(P_{crt}t_j+P_{Dj}\right)y_j+Q_{Dj}\right]=0$$

or

$$\left(P_{crt}t_0+P_{D0}\right)y_0+Q_{D0}+\sum_{j=1}^{n-1}\left[\left(P_{crt}t_j+P_{Dj}\right)y_j+Q_{Dj}\right]=0 \quad \text{……………………..……..……….. (69)}$$

or

$$Q_{D0}=-\left(P_{crt}t_0+P_{D0}\right)y_0-\sum_{j=1}^{n-1}\left[\left(P_{crt}t_j+P_{Dj}\right)y_j+Q_{Dj}\right]$$

Thus:

$$A_0=\frac{Q_{D0}}{R_0}=\frac{1}{R_0}\left\{-\left(P_{crt}t_0+P_{D0}\right)y_0-\sum_{j=1}^{n-1}\left[\left(P_{crt}t_j+P_{Dj}\right)y_j+Q_{Dj}\right]\right\} \quad \text{…………………….. (70)}$$

Thus $\left.\dfrac{dx}{dy}=\infty\right|_{@\,y=x=0} \rightarrow$ from Eq. 21 $B_0 = A_0$

And Eq. 24 does not apply. So rewriting Eq. 30 to

$$B_i^2 = A_i^2 + \frac{k_{i-1}^2}{k_i}\left[\left(y_{i-1} - y_i + A_{i-1}\right)^2 - B_{i-1}^2\right] \quad \text{.. (71)}$$

Therefore by substituting A_i , $A_{i\text{-}1}$, $B_{i\text{-}1}$ in Eq. 71 then B_i is found and $\phi 1_{i+1}$, $\phi 2_i$ and p_i are found from Eq. 25 and the ultimate buckling for P_{crt} is found for $\dot{x}_n = \infty$ or $\left.\dfrac{dy}{dx}=0\right|_{@\,y=y_n \text{ and } x=L}$ by replacing ϕ_0 and p_0 by ϕ_n and p_n. Also the ultimate buckling for P_{crt} can be found for $x = L$.

For Timoshenko linear buckling we rewrite Eq. 68 to

$$\frac{A_i}{B_i} = \frac{1}{\sqrt{1 + \dfrac{R_i}{R_{i-1}}\dfrac{EI_i}{EI_{i-1}}\left(1 - \dfrac{B_{i-1}^2}{\psi_{i-1}^2 A_{i-1}^2}\right)}} \quad \text{... (72)}$$

Thus by substituting $\dfrac{B_0}{A_0} = 1$ and ψ_0 from Eq. 62 with Eq. 70 all $\dfrac{A_i}{B_i}$ can be found from Eq. 72 and with Eq. 62 and 65 P_{crt} is found.

End Condition #2 Same as condition #1 except fixed at $x = L$ as in Fig 9.

From Eq. 24 we have:

$$B_{n-1}^2 = \left(y_{n-1} - A_{n-1}\right)^2 \quad \text{.. (73)}$$

For $g_n(y_n) = 0$ implies

$$\sum_{j=0}^{n-1}\left[\left(P_{crt}t_j + P_{Dj}\right)y_j + Q_{Dj}\right] = Q_{Dn}$$

or

$$\left(P_{crt}t_0 + P_{D0}\right)y_0 + Q_{D0} + \sum_{j=1}^{n-1}\left[\left(P_{crt}t_j + P_{Dj}\right)y_j + Q_{Dj}\right] = Q_{Dn} \quad \text{........................ (74)}$$

or

$$Q_{D0} = -\left(P_{crt}t_0 + P_{D0}\right)y_0 - \sum_{j=1}^{n-1}\left[\left(P_{crt}t_j + P_{Dj}\right)y_j + Q_{Dj}\right] + Q_{Dn}$$

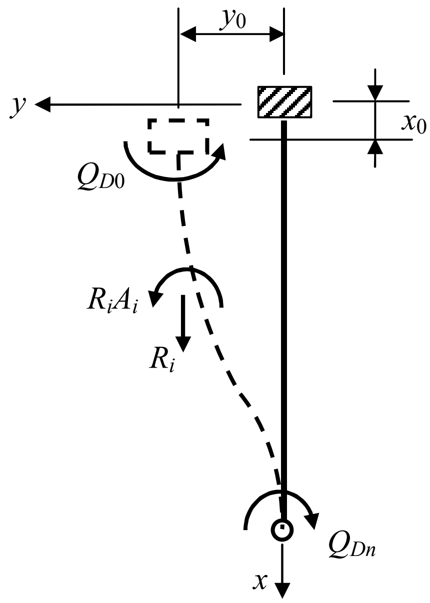

Fig. 9 Fixed – Rotational Fixed Column

Thus the procedure is to pick Q_{Dn} and find Q_{D0} from Eq. 74 and use procedure in condition #1 above to solve for y_i for given L_i and find all B_i then $\phi 1_{i+1}$, $\phi 2_i$ and p_i are found from Eq. 25 and the ultimate buckling for P_{crt} is found by replacing ϕ_0 and p_0 by ϕ_n and p_n. Then check if Eq. 73 is satisfied if not update Q_{Dn}.. Therefore, the ultimate buckling for P_{crt} can be found for $x = L$ as in Fig. 10.

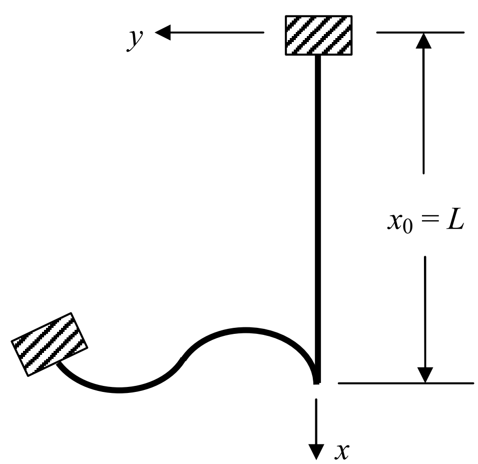

Fig. 10 – Ultimate Buckling

For Timoshenko linear buckling selected Q_{Dn} and find Q_{D0} from Eq. 74 and use procedure in condition #1 above to find all B_i then $\frac{A_i}{B_i}$ can be found from Eq. 72 and with Eq. 62 and 65 P_{crt} is found, then check if $B_{n-1} = A_{n-1}$ for $y_{n-1} \to 0$ is satisfied if not update Q_{Dn} until P_{crt} is found.

<u>End Condition #3</u> Pined both ends as in Fig 11.

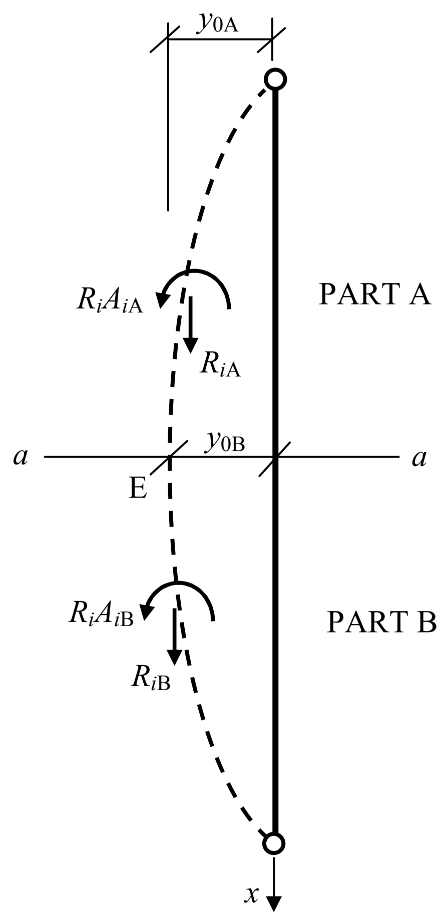

Fig 11 – Pined both ends.

Subdividing the column into two parts Part A and Part B at point E at line $a - a$ where the slope $\frac{dy}{dx} = 0$ in which it is to be found. Separate the loads of Part A and Part B and solve for P_{crt} for Part A using a straight cantilever column fixed at pint E and find y_{0A}. Then substitute P_{crt} in Part B using end condition #1 (pin at bottom end and rotational fixed at the top end but free to translate at the top.) and find y_{0B} . Check and see if $y_{0A} = y_{0B}$. If y_{0A} not equal to y_{0B} then move point E at line $a - a$ up if $y_{0A} > y_{0B}$ or down if $y_{0A} <$

y_{0B} until point E is found and P_{crt} is found. The ultimate buckling first criterion is when $\frac{dy}{dx} = \infty \rightarrow \frac{dx}{dy} = 0$ at point F or G and the second ultimate buckling when $x_{0A} + x_{0B} = 0$.
For Timoshenko linear buckling find P_{crt} for Part A at point E and P_{crt} for Part B separately and see if they are equal if not move point E until they are and P_{crt} is found.

End Condition #4 Fixed both ends.
This condition is similar to end condition #3 where Part A and B are as condition #2. The ultimate buckling criterion is only one for $x_{0A} + x_{0B} = 0$.

Applications:

One of the most useful applications is High Rise Buildings. We can take a one line column of several stories and introduce at the joints loads a $K_p y_i$ force and a $K_m \dot{y}_i$ moment where K_p and K_m is the stiffness of the adjacent members at the joist or from an output of a frame analysis program. By using the ultimate buckling or derive the linear buckling solution with springs the buckling load can be found. A prescribed drift for ultimate buckling will be needed. Another application is a truss tower with variable moment of inertia, such as crane towers, ski lift towers, gondolas towers, power utility towers, bridge towers, foundation towers and windmill towers etc. In these cases the ultimate buckling or linear buckling solution can be used depending on the applications.

Other applications of Buckling

1- Aerospace
2- Naval Architecture
3- Pipeline buckling in deep water applications
4- Offshore piles
5- Radio or Transmission Tower