Determining Active and Passive Pressures for Cohesion and Cohesionless Soils Using Variation for a Smooth Wall

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Abstract

Variational method is used to determine active and passive forces for a smooth wall with a cohesionless and cohesive back fill. The resulting slip surface shows that the extremum of the force occurs when the slip surface is the Coulomb line for a level backfill. The analysis shows that to achieve the Coulomb line the internal shear in the slices must be zero. For a sloped backfill with cohesion the resulting slip surface is not a line. Adding internal shear in the slices for the passive condition with a sloped backfill with a cohesiveless material is examined and compared with log-spiral method. For special boundary conditions, where the slip surface cannot be a line, the forces, the slip surface, the pressure on the wall and the location of the resultant on the wall are obtained for active and passive conditions. The method is adequate and will always give the derived forces and slip surfaces.

Introduction

The problem of active and passive earth pressure for a smooth wall has been solved by Coulomb theory (1776) [4], Rankine theory (1857) [8], and log-spiral theory after Terzaghi (1941 & 1943) [11,12]. These theories are presented in most soil mechanics'

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texts [12,13]. Each of these methods involves assuming a line slip surface as an approximation of a more complex surface as observed in both model tests and field observations. The reason in the differences between these slip surface methods and the laboratory is in the laboratory the observed slip surface is that of a slope stability problem and not of a wall pressure problem. In this paper, a more general solution to the problem is developed using variational methods using slices. In variational method, a general form of failure surface is derived, and then "varied" until the extreme condition of pressure on a wall is found. In a similar approach, variational method has been used to find the factor of safety in slope stability: Baker and Garber (1978) [1] and numerical methods by Leshechinsky (1990) [6]. The pressure discussed in this paper is for homogenous soil for a smooth wall. It can be readily extended to multilayer soil. However, further work is required. The limitation of the variational method is that it assumes a continuous function for the slip surface. This can be a problem for a heterogynous material as seen in Fig. 1. The method fails and

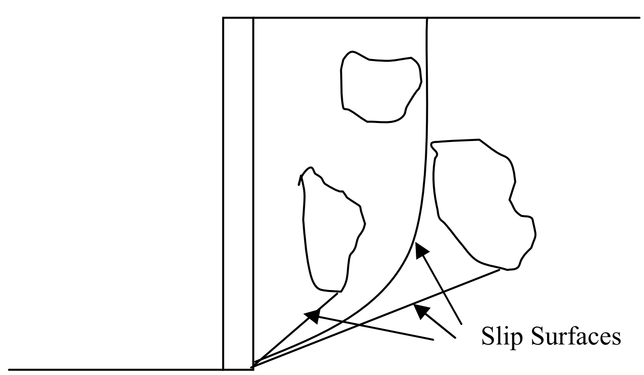


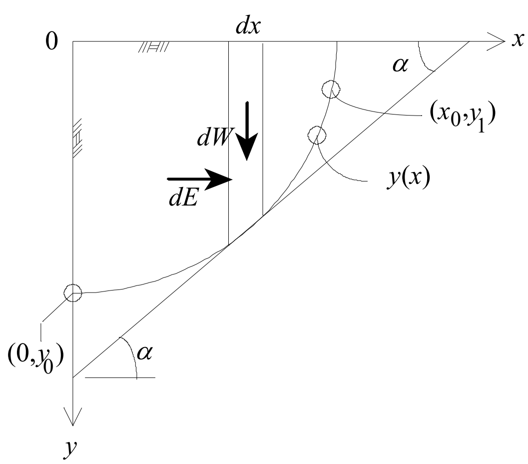
FIG. 1 - Heterogynous Material with Slip Surfaces

revisions are needed, possibly piecewise functions are needed. Note: piesewise continous function will be needed for multilayer homogenous soil. The general methodology has been classically to find the extremum for the earth pressure by finding a slip surface where a potentially soil failure would occur for extremum pressure. Thus, the design has captured the worst condition for the force on the wall. Historically, this method has been used satisfactorily provided the material properties have been reached. It is impossible in practice to test for all the soil friction values and cohesion through borings logs, thus engineering judgment has to be made for these values. In order to use the method proposed properly and interpret the results, care has to be taken in selecting the material properties.

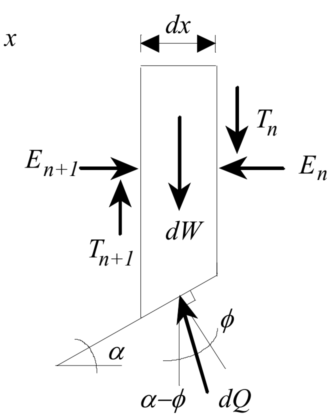
With the variational method, [15], one selects arbitrary admissible slip surfaces and determines the forces acting on the boundary of the earth mass. The definitive slip surface is one that yields an extremum value for the earth pressure. In this paper, variational methods are used to determine the slip surface, based on some practical assumptions. The solution shows that the Coulomb wedge is a particular case of the general solution presented here. It is believed that the application of variational methods to earth pressure problems is a practical advancement to understanding of retaining wall problems. Five different analyses will be done to avoid giving one cumbersome general equation and give a closed form solution for the special cases. In analysis 1 a closed form solution is reached, it is assumed a level backfill for cohesive and cohesiveless material, which is useful for most site conditions. Analysis 2 is for a sloped backfill and slanted wall for cohesiveless material, which does not include cohesion to give a closed form solution. Analysis 3 is repeating analysis 2 with cohesion where the pressure is not given in a closed form but can be obtained numerically. In analysis 4 a closed form solution is reached for passive earth pressure condition for sloped backfill with cohesiveless material to include internal vertical shear from the slope backfill. Analysis 5 is a repeat of Analysis 4 with cohesion.

Analysis 1

In this analysis, it is desired not to restrict the slip surface to a line and to find a function $y(x)$ that would extremize E, as shown in Fig. 2(a). It is assumed that the wall moves



Proposed Slip Surface
FIG. 2(a)



A Slice
FIG. 2(b)

FIG. 2 – Slip Surface for Level Backfill

sufficiently such that the friction between the wedges equal 0. Thus, $T_{n+1} = T_n$, as in Fig. 2(b), for all the wedges. Since the wall has no friction, $T_i = 0$ for all the wedges. This assumption is similar to the assumptions used in stability analysis by the method of slices (Bishop (1955) [2]) and in analysis of passive earth pressures for a wall with friction by Shields and Tolunay (1973) [10]. If the T_i 's are to be included in the derivations, K_0 can be determined using incipient shear method. Thus, each slice is in equilibrium, so the overturning moment remains in balance and not of concern.

Thus, the force dE can be written as

$$E_{n+1} - E_n = dE = \tan(\alpha - \phi) dW - c\, ds[\cos\alpha + \sin\alpha \tan(\alpha - \phi)] \qquad (1)$$

Replacing dW by $\gamma y dx$ and ds by $dx/\cos\alpha$ and integrating yields

$$E = \int_0^{x_0} \{\tan(\alpha - \phi)\gamma y - c[1 + \tan\alpha \tan(\alpha - \phi)]\} dx \qquad (2)$$

or

$$E = \int_0^{x_0} \left\{ \frac{\tan\alpha - \tan\phi}{1 + \tan\alpha \tan\phi} \gamma y - c\left[1 + \frac{\tan\alpha(\tan\alpha - \tan\phi)}{1 + \tan\alpha \tan\phi}\right]\right\} dx \qquad (3)$$

Now $\tan\alpha = -\dfrac{dy}{dx} = y'$ (4)

Substituting Eq. 4 in Eq. 3 yields

$$E = \int_0^{x_0} \left\{ \frac{-y' - \tan\phi}{1 - y' \tan\phi} \gamma y - c\left[1 - \frac{y'(-y' - \tan\phi)}{1 - y' \tan\phi}\right]\right\} dx \qquad (5)$$

Eq. 5 can written as:

$$E = \int_0^{x_0} \left\{ \left[\cot\phi - \frac{\tan\phi + \cot\phi}{1 - y' \tan\phi} \right] \gamma y - c \left[1 - \frac{1}{\sin^2\phi} - \cot\phi \, y' + \frac{1}{\sin^2\phi (1 - y' \tan\phi)} \right] \right\} dx$$

……………………………………… (6)

It is necessary to find the extremum of E in Eq. 6 for the boundary condition:

(**i**) Given $(0, y_0)$ and (x_0, y_1), find the slip surface that passes through these points.

(**ii**) Given $(0, y_0)$ and P_0 located somewhere on a given line $x = x_0$ or a line $y = y_1$, find the slip surface.

Therefore Euler's equation [15] applies where

$$\Re = \left[\cot\phi - \frac{\tan\phi + \cot\phi}{1 - y' \tan\phi} \right] \gamma y - c \left[1 - \frac{1}{\sin^2\phi} - \cot\phi \, y' + \frac{1}{\sin^2\phi (1 - y' \tan\phi)} \right]$$ .... (7)

and $\frac{\partial \Re}{\partial y} - \frac{d}{dx}\left(\frac{\partial \Re}{\partial y'} \right) = 0$

which can be written as $\Re - y' \frac{\partial \Re}{\partial y'} = h_0$ .... (8)

where $\Re$ does not involve x explicitly, and h_0 is a constant.

Substituting in Eq. 8 yields

$$\left[\cot\phi - \frac{\tan\phi + \cot\phi}{1 - y'\tan\phi} + \frac{y'}{\cos^2\phi(1 - y'\tan\phi)^2}\right]\gamma y$$

$$- c\left[-\cot^2\phi + \frac{1}{\sin^2\phi(1 - y'\tan\phi)} - \frac{\tan\phi y'}{\sin^2\phi(1 - y'\tan\phi)^2}\right] = h_0$$

.... (9)

Eq. 9 reduces to the following:

$$\left(y'^2 + 2y'\tan\phi - 1\right)\left(\gamma y\tan\phi + c\right) = h_0\left(1 - y'\tan\phi\right)^2 \quad \text{.. (10)}$$

or

$$\left(y' + \tan\phi + 1/\cos\phi\right)\left(y' + \tan\phi - 1/\cos\phi\right)\left(\gamma y\tan\phi + c\right) = h_0\left(1 - y'\tan\phi\right)^2 \quad \text{.................. (11)}$$

Note Eq. 10 is a parabola in y'. From Eq. 11: $h_0 \geq 0$ for $y' \leq -\tan\phi - 1/\cos\phi$ or $y' \geq -\tan\phi + 1/\cos\phi$, and $h_0 < 0$ for $-\tan\phi - 1/\cos\phi \leq y' \leq -\tan\phi + 1/\cos\phi$. From trigonometric identity

$$\tan\left(\frac{\pi}{4} \pm \frac{\phi}{2}\right) = \tan\left(\frac{\pi/2 \pm \phi}{2}\right) = \frac{1 - \cos(\pi/2 \pm \phi)}{\sin(\pi/2 \pm \phi)} = \frac{1 \pm \sin\phi}{\cos\phi} = \pm\tan\phi + \frac{1}{\cos\phi} \quad \text{......... (12)}$$

From Eq. 11 and Eq. 12, $h_0 \geq 0$ for $\alpha \geq \pi/4 + \phi/2$ or $\alpha \leq -\left(\pi/4 - \phi/2\right)$, and $h_0 < 0$ for $-\left(\pi/4 - \phi/2\right) \leq \alpha \leq \pi/4 + \phi/2$. Taking α to be positive yields the following bound on h_0:

$h_0 \geq 0$ for $\alpha \geq \pi/4 + \phi/2$, and $h_0 < 0$ for $0 \leq \alpha \leq \pi/4 + \phi/2$ ......Active............. (13)

For the passive condition, replacing ϕ by $-\phi$ and c by $-c$ in Eq. 11 yields

$h_0 \geq 0$ for $\alpha \geq \pi/4-\phi/2$, and $h_0 < 0$ for $0 \leq \alpha \leq \pi/4-\phi/2$ ......Passive............ (14)

The slip surface can be derived by rewriting Eq. 10 to

$$\left(y-h_1\tan^2\phi\right)y'^2+2\tan\phi\left(y+h_1\right)y'-\left(y+h_1\right)=0 \quad \text{...} \quad (15)$$

where $h_1=\dfrac{h}{1+\dfrac{c}{\gamma y\tan\phi}}$ and $h=\dfrac{h_0}{\gamma\tan\phi}$ is a new constant. Rewriting Eq. 15 in

x' instead of y' yields (See Appendix I)

$$x'=\tan\phi\pm\frac{1}{\cos\phi}\sqrt{\frac{y+\dfrac{c}{\gamma\tan\phi}}{y+\dfrac{c}{\gamma\tan\phi}+h}} \quad \text{...} \quad (16)$$

If $h = 0$ in Eq. 16, then x' becomes a constant and $y(x)$ must follow the Coulomb wedge.

Since $x'=-\cot\alpha$ and $\alpha=\pi/4+\phi/2$ for a Coulomb wedge it follows from Eq. 12 that

$$x'=-\cot(\pi/4+\phi/2)=-\tan(\pi/4-\phi/2)=\tan\phi-1/\cos\phi \quad \text{.................................} \quad (17)$$

In order to satisfy Eq. 17, the plus sign in Eq. 16 can be dropped, and the equations become

$$x' = \tan\phi - \frac{1}{\cos\phi}\sqrt{\frac{y + \dfrac{c}{\gamma\tan\phi}}{y + \dfrac{c}{\gamma\tan\phi} + h}} \quad \text{.............Active...} \quad (18a)$$

For the passive condition, replacing ϕ by $-\phi$ and c by $-c$

$$x' = -\tan\phi - \frac{1}{\cos\phi}\sqrt{\frac{y + \dfrac{c}{\gamma\tan\phi}}{y + \dfrac{c}{\gamma\tan\phi} + h}} \quad \text{.............Passive...} \quad (18b)$$

Integrating Eq. 18a yields the slip surface equation

$$x = y\tan\phi - \frac{1}{\cos\phi}\left(\sqrt{\left(y + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y + \frac{c}{\gamma\tan\phi}\right)} - \frac{h}{2}\ln\left|2\sqrt{\left(y + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y + \frac{c}{\gamma\tan\phi}\right)} + 2\left(y + \frac{c}{\gamma\tan\phi}\right) + h\right|\right) + k$$

.... (19)

where k is the constant of integration. Substituting the point at $x = 0$ $y = y_0$ yields

$$k = -y_0 \tan\phi + \frac{1}{\cos\phi}\left(\sqrt{\left(y_0 + \frac{c}{\gamma \tan\phi}\right)^2 + h\left(y_0 + \frac{c}{\gamma \tan\phi}\right)} - \frac{h}{2}\ln\left|2\sqrt{\left(y_0 + \frac{c}{\gamma \tan\phi}\right)^2 + h\left(y_0 + \frac{c}{\gamma \tan\phi}\right)} + 2\left(y_0 + \frac{c}{\gamma \tan\phi}\right) + h\right|\right) + k$$

.... (20)

and the active slip surface, Eq. 19, becomes

$$x = (y - y_0)\tan\phi - \frac{1}{\cos\phi}\left(\sqrt{\left(y + \frac{c}{\gamma \tan\phi}\right)^2 + h\left(y + \frac{c}{\gamma \tan\phi}\right)} - \sqrt{\left(y_0 + \frac{c}{\gamma \tan\phi}\right)^2 + h\left(y_0 + \frac{c}{\gamma \tan\phi}\right)} - \frac{h}{2}\ln\left|\frac{2\sqrt{\left(y + \frac{c}{\gamma \tan\phi}\right)^2 + h\left(y + \frac{c}{\gamma \tan\phi}\right)} + 2\left(y + \frac{c}{\gamma \tan\phi}\right) + h}{2\sqrt{\left(y_0 + \frac{c}{\gamma \tan\phi}\right)^2 + h\left(y_0 + \frac{c}{\gamma \tan\phi}\right)} + 2\left(y_0 + \frac{c}{\gamma \tan\phi}\right) + h}\right|\right)$$

……………………….. (21a)

For passive it becomes

$$x = -(y - y_0)\tan\phi - \frac{1}{\cos\phi}\left(\sqrt{\left(y + \frac{c}{\gamma \tan\phi}\right)^2 + h\left(y + \frac{c}{\gamma \tan\phi}\right)} - \sqrt{\left(y_0 + \frac{c}{\gamma \tan\phi}\right)^2 + h\left(y_0 + \frac{c}{\gamma \tan\phi}\right)} - \frac{h}{2}\ln\left|\frac{2\sqrt{\left(y + \frac{c}{\gamma \tan\phi}\right)^2 + h\left(y + \frac{c}{\gamma \tan\phi}\right)} + 2\left(y + \frac{c}{\gamma \tan\phi}\right) + h}{2\sqrt{\left(y_0 + \frac{c}{\gamma \tan\phi}\right)^2 + h\left(y_0 + \frac{c}{\gamma \tan\phi}\right)} + 2\left(y_0 + \frac{c}{\gamma \tan\phi}\right) + h}\right|\right)$$

……………...………………..(21b)

To find the active force *E,* Eq. 5 can be rewritten in terms of x' instead of y',

dy instead of dx, and the interval $[y_0, 0]$ instead of $[0, x_0]$, where y_1 is taken to be 0 in Fig. 2(a). Thus

$$E = \int_{y_0}^{0} \left\{ \frac{1 + x' \tan\phi}{\tan\phi - x'} \gamma x' y + c \left[\frac{1 + (x')^2}{\tan\phi - x'} \right] \right\} dy \qquad (22)$$

Substituting Eq. 18a in Eq. 22 and rearranging yields

$$E = \int_0^{y_0} \gamma y \left(\tan^2\phi + \frac{1}{\cos^2\phi} - \frac{\tan\phi}{\cos\phi} \frac{2\left(y + \frac{c}{\gamma\tan\phi}\right) + h}{\sqrt{\left(y + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y + \frac{c}{\gamma\tan\phi}\right)}} \right) dy \\ - c\int_0^{y_0} \left(2\tan\phi - \frac{1}{\cos\phi} \frac{2\left(y + \frac{c}{\gamma\tan\phi}\right) + h}{\sqrt{\left(y + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y + \frac{c}{\gamma\tan\phi}\right)}} \right) dy \qquad (23)$$

Thus the mathematical pressure $q(y) = \frac{dE}{dy}$, from each slice to another, becomes

$$q(y) = \frac{dE}{dy} = \gamma y \left(\tan^2\phi + \frac{1}{\cos^2\phi} - \frac{\tan\phi}{\cos\phi} \frac{2\left(y + \frac{c}{\gamma\tan\phi}\right) + h}{\sqrt{\left(y + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y + \frac{c}{\gamma\tan\phi}\right)}} \right) \\ - c \left(2\tan\phi - \frac{1}{\cos\phi} \frac{2\left(y + \frac{c}{\gamma\tan\phi}\right) + h}{\sqrt{\left(y + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y + \frac{c}{\gamma\tan\phi}\right)}} \right)$$

.... (24)

It is desired to investigate the extremum of $q(y)$ with respect to h, thus

$$\frac{dq}{dh} = 0 \text{ yields } h = 0 \quad \text{.......... (25)}$$

Thus the extremum occurs at the Coulomb wedge for passive and active. Hence it has been shown that the Coulomb wedge is the wedge for a smooth wall that moves sufficiently such that $T_{n+1} = T_n$. It is noted that all of $T_i = 0$ since the wall is smooth. For other prescribed conditions (**i**) and (**ii**) above it is necessary to use the slip surface of Eqs. 21a-b. Integrating Eq. 23 yields the active force

$$E = \gamma \frac{y_0^2}{2}\left(\tan^2\phi + \frac{1}{\cos^2\phi}\right) + 2cy_0\tan\phi$$
$$-\gamma\frac{\tan\phi}{\cos\phi}\left(\left(y_0 + \frac{c}{\gamma\tan\phi} - \frac{h}{2}\right)\sqrt{\left(y_0 + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y_0 + \frac{c}{\gamma\tan\phi}\right)} - \left(\frac{c}{\gamma\tan\phi} - \frac{h}{2}\right)\sqrt{\left(\frac{c}{\gamma\tan\phi}\right)^2 + h\left(\frac{c}{\gamma\tan\phi}\right)}\right)$$
$$-\gamma\frac{\tan\phi}{\cos\phi}\left(\frac{h^2}{4}\right)\ln\left|\frac{2\sqrt{\left(y_0 + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y_0 + \frac{c}{\gamma\tan\phi}\right)} + h + 2\left(y_0 + \frac{c}{\gamma\tan\phi}\right)}{2\sqrt{\left(\frac{c}{\gamma\tan\phi}\right)^2 + h\left(\frac{c}{\gamma\tan\phi}\right)} + h + \frac{2c}{\gamma\tan\phi}}\right|$$

…………………………….......... (26a)

and the passive force

$$E=\gamma\frac{y_0^2}{2}\left(\tan^2\phi+\frac{1}{\cos^2\phi}\right)+2cy_0\tan\phi$$

$$+\gamma\frac{\tan\phi}{\cos\phi}\left(\left(y_0+\frac{c}{\gamma\tan\phi}-\frac{h}{2}\right)\sqrt{\left(y_0+\frac{c}{\gamma\tan\phi}\right)^2+h\left(y_0+\frac{c}{\gamma\tan\phi}\right)}-\left(\frac{c}{\gamma\tan\phi}-\frac{h}{2}\right)\sqrt{\left(\frac{c}{\gamma\tan\phi}\right)^2+h\left(\frac{c}{\gamma\tan\phi}\right)}\right)$$

$$+\gamma\frac{\tan\phi}{\cos\phi}\left(\frac{h^2}{4}\right)\ln\left|\frac{2\sqrt{\left(y_0+\frac{c}{\gamma\tan\phi}\right)^2+h\left(y_0+\frac{c}{\gamma\tan\phi}\right)}+h+2\left(y_0+\frac{c}{\gamma\tan\phi}\right)}{2\sqrt{\left(\frac{c}{\gamma\tan\phi}\right)^2+h\left(\frac{c}{\gamma\tan\phi}\right)}+h+\frac{2c}{\gamma\tan\phi}}\right|$$

………………………………........... (26b)

Note: that the right hand term involving h in Eqs. 26a-b has a minimum at h=0. It is desired to show that for $h<-\frac{c}{\gamma\tan\phi}$ the slip surface does not reach the top of ground at the point (x_0,0). It has been shown in Eqs. 13 and 14 that if $h_0 < 0$ (or $h < 0$), then $\alpha\le\pi/4+\phi/2$ for active, and $\alpha\le\pi/4-\phi/2$ for passive. Thus a slip surface with $h<-\frac{c}{\gamma\tan\phi}$ is further than the triangle Coulomb wedge as shown in Fig. 3.

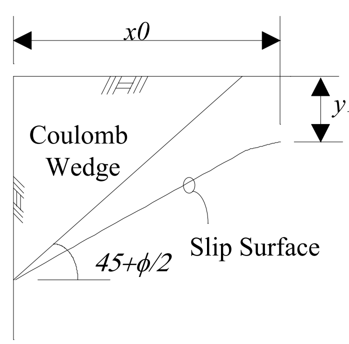


FIG 3 – Slip Surface for Level Backfill with Constraint $h < 0$

Furthermore, Eq. 18a-b shows that $x' = \infty$ or $y' = 0$ at some value $y = -h - \dfrac{c}{\gamma \tan\phi}$.

Since $h < -\dfrac{c}{\gamma \tan\phi}$, the y value must be positive and cannot be 0. Thus y is below ground. Also, for $y < -h - \dfrac{c}{\gamma \tan\phi}$ the term $\sqrt{y + \dfrac{c}{\gamma \tan\phi} + h}$ is not defined in Eq. 18a-b.

Thus for $h < -\dfrac{c}{\gamma \tan\phi}$ Eqs. 26a-b are not applicable.

Eq. 22 must be integrated from y_0 to y_1 instead of y_0 to 0, where the point (x_0, y_1) is a point underground. This integration yields

$$
\begin{aligned}
E = {} & \gamma\left(\frac{y_0^2}{2} - \frac{y_1^2}{2}\right)\left(\tan^2\phi + \frac{1}{\cos^2\phi}\right) + 2c(y_0 - y_1)\tan\phi \\
& - \gamma\frac{\tan\phi}{\cos\phi}\left(\left(y_0 + \frac{c}{\gamma\tan\phi} - \frac{h}{2}\right)\sqrt{\left(y_0 + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y_0 + \frac{c}{\gamma\tan\phi}\right)}\right. \\
& \qquad\qquad \left. - \left(y_1 + \frac{c}{\gamma\tan\phi} - \frac{h}{2}\right)\sqrt{\left(y_1 + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y_1 + \frac{c}{\gamma\tan\phi}\right)}\right) \\
& - \gamma\frac{\tan\phi}{\cos\phi}\left(\frac{h^2}{4}\right)\ln\left|\frac{2\sqrt{\left(y_0 + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y_0 + \frac{c}{\gamma\tan\phi}\right)} + h + 2\left(y_0 + \frac{c}{\gamma\tan\phi}\right)}{2\sqrt{\left(y_1 + \frac{c}{\gamma\tan\phi}\right)^2 + h\left(y_1 + \frac{c}{\gamma\tan\phi}\right)} + h + 2\left(y_1 + \frac{c}{\gamma\tan\phi}\right)}\right|
\end{aligned}
$$

..................................Active.................................... (27a)

$$E=\gamma\left(\frac{y_0^2}{2}-\frac{y_1^2}{2}\right)\left(\tan^2\phi+\frac{1}{\cos^2\phi}\right)+2c(y_0-y_1)\tan\phi$$

$$+\gamma\frac{\tan\phi}{\cos\phi}\left(\left(y_0+\frac{c}{\gamma\tan\phi}-\frac{h}{2}\right)\sqrt{\left(y_0+\frac{c}{\gamma\tan\phi}\right)^2+h\left(y_0+\frac{c}{\gamma\tan\phi}\right)}\right.$$

$$\left.-\left(y_1+\frac{c}{\gamma\tan\phi}-\frac{h}{2}\right)\sqrt{\left(y_1+\frac{c}{\gamma\tan\phi}\right)^2+h\left(y_1+\frac{c}{\gamma\tan\phi}\right)}\right)$$

$$+\gamma\frac{\tan\phi}{\cos\phi}\left(\frac{h^2}{4}\right)\ln\left|\frac{2\sqrt{\left(y_0+\frac{c}{\gamma\tan\phi}\right)^2+h\left(y_0+\frac{c}{\gamma\tan\phi}\right)}+h+2\left(y_0+\frac{c}{\gamma\tan\phi}\right)}{2\sqrt{\left(y_1+\frac{c}{\gamma\tan\phi}\right)^2+h\left(y_1+\frac{c}{\gamma\tan\phi}\right)}+h+2\left(y_1+\frac{c}{\gamma\tan\phi}\right)}\right|$$

.................................Passive................................. (27b)

Thus it can be easily concluded that $h = 0$ is the only minimum of the h terms in Eqs. 26a-b. Hence E has a maximum at $h = 0$ for active and a minimum at $h = 0$ for passive. This shows that the Coulomb wedge gives the extremum for E as it has been shown in Eq. 25.

Condition (i)

It is seen that Eqs. 18a-b , 21a-b , 26a-b , and 27a-b are useful for condition (**i**) when the slip surface must pass through a point and physically is not represented by the Coulomb wedge. The following examples have this situation:

Example 1 Consider Fig. 4, where the passive pressure is to be calculated for a slip

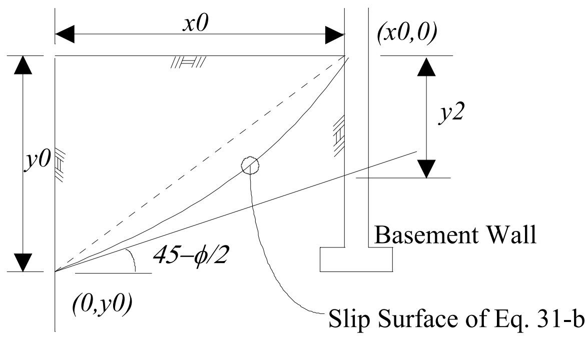


FIG. 4 – Slip Surface for Passive with and Existing Basement Constrained Example 1

surface that is controlled by the presence of a basement wall. The Coulomb wedge cannot physically go through the wall. Thus the general slip surface is applicable.

In this situation y_0, x_0 are known. Thus, substituting their values in Eq. 21b yields

$$x_0 = y_0 \tan\phi + \frac{1}{\cos\phi}\left(\sqrt{{y_0}^2 + hy_0} + \frac{h}{2}\ln\left|\frac{h}{2\sqrt{{y_0}^2 + hy_0} + 2y_0 + h}\right|\right) \quad \text{.................................} \quad (28)$$

The value h can be obtained numerically from Eq. 28, and E can be calculated from Eq. 26b. Thus, if γ = 120 pcf (1.93 g/cm³), ϕ = 30 degrees, c = 0, y_0 = 10 ft (3.05 m), and x_0 = 10 ft (3.05 m), then from Eq. 28, h = 27.3318 ft (8.3307 m). Taking $\lambda = h / y_0 = 2.73321$ in Eq. 26b gives E = 21,455 plf (31.97 g/m). If a straight line is used from (0, y_0) to (x_0, 0), then α = 45 degrees. Evaluating the Coulomb wedge $y = y_0$ = 10 ft (3.05 m) gives E = 22,392 plf (33.37 g/m). Note that Eq. 26b gives the extremum with 4.4% difference over the straight line. Note: If treated as the full wedge with K_p = 3 where y = 5.77ft (1.76 m) is triangular pressure and 4.33 ft (1.29 m) is pressure with surcharge then E = 18,000 pfl (26.82 g/m). In this situation it is assumed the wall is rigid thus the slip surface is a curve and E = 21,455 plf (31.97 g/m).

Example 2 Consider Fig. 5, where a slip surface passing through $(0, y_0)$ to (x_0, y_1) must

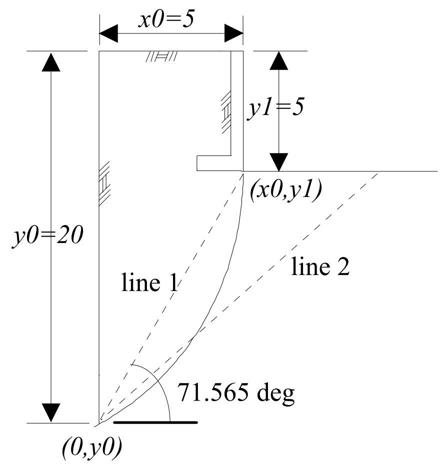
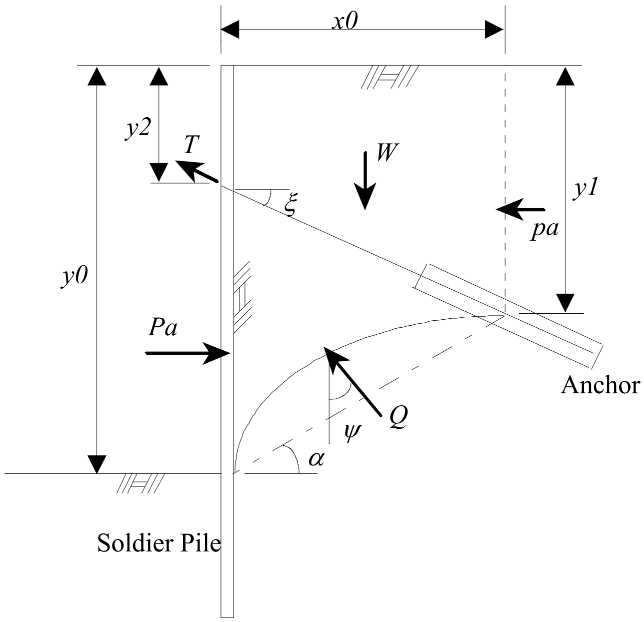


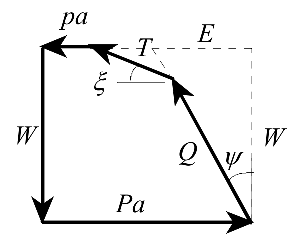
FIG. 5 – Slip Surface for Active with and Existing Basement Constrained Example 2

be analyzed for an active condition. $\gamma = 120$ pcf (1.93 g/cm³), and $\phi = 30$ degrees, $c = 0$. From Eq. 21a substituting $y_0 = 20$ ft (6.1 m), $x = x_0 = 5$ ft (1.52 m), $y = y_1 = 5$ ft (1.52 m), and calculating numerically $h = 6.88284$ ft (2.0979 m). Substituting in Eq. 27a gives E = 6,740 plf (10.05 g/m). If comparing with a straight line, line 1, from $(0, y_0)$ to (x_0, y_1), it gives E = 6,651 plf (9.91 g/m), where the Coulomb wedge was used with W = .5(5+20)(5)(120)=7,500 plf (11.18 g/m), and $\alpha = 71.565$ degrees. Thus, a 1.3% difference over the derived is obtained. To find E_{max} further analysis must be done for line 2 and the result must be compared with line 1.

Example 3 In the case of stability analysis for tieback wall, as shown in Fig. 6(a) , the



Tieback Wall
FIG. 6-a



Force Diagram
FIG. 6-b

FIG. 6 - Slip Surface to Determine Safety Factor of a one Tieback Wall

slip surface is shown to pass through the point in the middle of the anchor, see reference [3,7,9,16]. In this case the stability factor T_{max}/T_{design} is to be calculated as in reference [16]. From the force diagram, Fig. 6(b), the summation of horizontal and vertical forces are

$$p_a + T\cos\xi + Q\sin\psi = P_a \quad (29)$$

$$T\sin\xi + Q\cos\psi = W \quad (30)$$

Substituting Q from Eq. 30 in Eq. 29 yields

$$T = \frac{P_a - W\tan\psi - p_a}{\cos\xi - \sin\xi\tan\psi} \quad \text{(31)}$$

From Eq. 31 $T = T_{max}$ can be calculated where $W\tan\psi = E$ of Eq. 27a, and the constant h can be found from Eq. 21a for a given (x_0, y_1). W can be calculated numerically from integrating the right hand side of Eq. 21a from y_1 to y_0

$$W = \gamma y_1 x_0 + \gamma \int_{y_1}^{y_0} f(y)dy \quad \text{(32)}$$

where $f(y)$ is the right hand side of Eq. 21a. Thus, $\psi = \tan^{-1}\left(\frac{E}{W}\right)$,

$p_a = \frac{1}{2}\gamma y_1^{\,2}\tan^2\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$, and $P_a = \frac{1}{2}\gamma y_0^{\,2}\tan^2\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$.

Taking for example y_0 = 20 ft (6.1 m), γ = 120 pcf (1.93 g/cm^3), ϕ = 30 degrees, c = 0, ξ = 20 degrees, T_{design} = 3,872 plf (5.77 g/m), P_a = .333(120)(20)(20)/2 = 8,000 plf (11.92 g/m), y_2 = 6 ft (1.83 m), x_0 = 15 ft (4.57 m), y_1 = 6 + 15 tan(20) = 11.46 ft (3.49 m), p_a = .333(120)(11.46)(11.46)/2 = 2,626 plf (3.91 g/m), and substituting $x = x_0$ = 15 ft (4.57 m), and $y = y_1$ = 11.46 ft (3.49 m) in Eq. 21a gives h = -10.8507 ft (-3.3073 m). Substituting in Eq. 27a gives E = 154.4 plf (0.23 g/m). Integrating Eq. 32 numerically gives W = 26,802 plf (39.94 g/m). Thus, ψ = 0.3301 degrees, and T in Eq. 31 becomes 5,566 plf (8.3 g/m). The stability factor T_{max}/T_{design} = 5,566/3,872 = 1.438. If using a straight line method, where ψ in Eq. 31 becomes $\alpha-\phi$, α = 29.66 degrees, and W = $0.5\gamma(y_0+y_1)x_0$ = 28,314 plf (42.2 g/m), then T = 5,885 plf (8.77 g/m), and the stability factor = 5,885/3,872 = 1.52. This gives 6% difference in the safety factor over the derived method.

Condition (ii)

For condition (**ii**), where the slip surface must pass through a line at x_0 or at y_1 below x_0 , it yields $\left.\frac{\partial \Re}{\partial y'}\right|_{x=x_0}$ = 0, or from Eq. 7

$$-\gamma \frac{y_1}{\cos^2 \phi (1 - y' \tan \phi)^2} - c \frac{\tan \phi}{\sin^2 \phi (1 - y' \tan \phi)^2} + c \cot \phi = 0$$

or .... (33)

$$\frac{y_1 + \dfrac{c}{\gamma \tan \phi}}{(1 - y' \tan \phi)^2} - \frac{c \cot \phi \cos^2 \phi}{\gamma} = 0$$

Substituting Eq. 18a with y_1 instead of y in Eq. 33 where $x' = 1 / y'$ yields

$$\left(\tan \phi \sqrt{y_1 + \frac{c}{\gamma \tan \phi} + h} - \frac{1}{\cos \phi} \sqrt{y_1 + \frac{c}{\gamma \tan \phi}} \right)^2 - \frac{c}{\gamma \tan \phi} = 0 \quad \text{............active....... (34a)}$$

$$\left(- \tan \phi \sqrt{y_1 + \frac{c}{\gamma \tan \phi} + h} - \frac{1}{\cos \phi} \sqrt{y_1 + \frac{c}{\gamma \tan \phi}} \right)^2 - \frac{c}{\gamma \tan \phi} = 0 \quad \text{.........passive....... (34b)}$$

Thus

$$y_1 = h\tan^2\phi + 2\tan^2\phi\left(\frac{c}{\gamma\tan\phi}\right) \pm 2\frac{\tan\phi}{\cos\phi}\sqrt{\left(\frac{c}{\gamma\tan\phi}\right)^2 + h\left(\frac{c}{\gamma\tan\phi}\right)}$$

.... (35)

At $\dot{y} \to \infty \quad \dot{x} \to 0$ then

$$y_2 = h\tan^2\phi - \frac{c}{\gamma\tan\phi}$$ ....only active .... (35a)

Substitute Eq. 35 in Eq. 21a and 21b for a given x_0 and find h. If y_1 is negative then $y_1 = 0$ and h can be calculated from Eq. 21a and 21b for a given x_0. If y_1 is above y_2 then y_2 controls and find h from Eq. 21a and 21b. Note: y_1 is zero only if $h = 0$ and $c = 0$. Thus, it can be obtained from Eq. 34a is the following problem: Consider that the slip surface must pass through a line at $x = x_0$ and have $y_1 = h\tan^2\phi$ below x_0 and $c = 0$. This makes $x' = 0$ or $y' = \infty$ in Eq. 18a. This condition can occur in the real world as seen in example 4 below, and E can be obtained from Eq. 27a. Note $y_1 = h\tan^2\phi$ with $c = 0$ is invalid for passive pressure, since Eq. 44b is not zero. For passive Eq. 45 is not valid when replacing ϕ by $-\phi$ and c by $-c$ and the slip surface must pass through the point (x0, 0) when there is a rigid obstacle otherwise it can be analysed as a surcharge problem as seen in example 1.

For $y_1 = -h - \dfrac{c}{\gamma\tan\phi}$, $h < 0$ and $y' = 0$ or $x' = \infty$ in Eq. 18a-b. This situation can happen if the slip surface must pass through a line $x = x_0$ where the slope must be zero at y_1 below x_0 . Fig. 7 shows such an example of an passive condition with $c = 0$, where the concrete

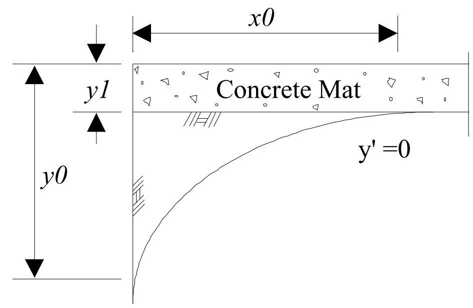


FIG. 7 – Slip Surface for Passive with Mat Slab Constrained Special Condition

mat shown on top can sustain itself. Thus at the line $x = x_0$, $y' = 0$ due to deformation. In this example the weight of the slab is taken to be the same as the weight of soil.

Substituting $c = 0$, $h = -y_1$ and $y = y_1$ in Eq. 21b yields

$$x_0 = -(y_1 - y_0)\tan\phi + \frac{1}{\cos\phi}\left(\sqrt{y_0^{\,2} - y_1 y_0} - \frac{y_1}{2}\ln\left|\frac{y_1}{2\sqrt{y_0^{\,2} - y_1 y_0} + 2y_0 - y_1}\right|\right) \text{ .... (36)}$$

and the passive force E can be obtained from Eq. 27b:

$$E = \gamma\left(\frac{y_0^{\,2}}{2} - \frac{y_1^{\,2}}{2}\right)\left(\tan^2\phi + \frac{1}{\cos^2\phi}\right)$$

$$+\frac{\gamma\tan\phi}{\cos\phi}\left[\left(y_0 + \frac{y_1}{2}\right)\sqrt{y_0^{\,2} - y_1 y_0} + \frac{y_1^{\,2}}{4}\ln\left|\frac{2\sqrt{y_0^{\,2} - y_1 y_0} + 2y_0 - y_1}{y_1}\right|\right]$$

.... (37)

Example 4 Consider Fig. 8, where the active pressure is to be calculated for a slip surface

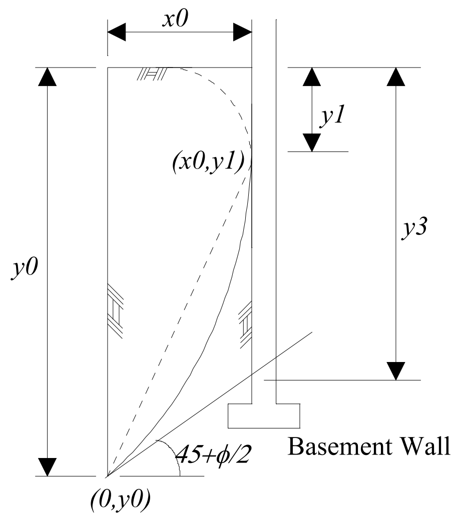


FIG. 8 – Slip Surface for Active with a Neighboring Structure Example 4

that avoids the basement wall with $c = 0$. This problem is also similar to finding the active pressure for a wall adjacent to rocks.

From Eq. 35

$$h = y_1 \cot^2 \phi \text{ .... (38)}$$

Substituting Eq. 38 in Eq. 21a yields

$$\frac{x_0}{y_0} = (z-1)\tan\phi - \frac{1}{\cos\phi}\left(\frac{z}{\sin\phi} - \sqrt{1+z\cot^2\phi} - \frac{z\cot^2\phi}{2}\ln\left|\frac{(2/\sin\phi + 2 + \cot^2\phi)z}{2\sqrt{1+z\cot^2\phi} + 2 + z\cot^2\phi}\right|\right)$$

.... (39)

where $z = y_1/y_0$. Thus, for a given x_0 and y_0 , z can be calculated numerically from Eq. 39. Taking $h = y_1 \cot^2\phi = zy_0 \cot^2\phi$, it can be substituted in Eq. 27a to give the active force. Taking for example y_0 = 20 ft (6.1 m), γ = 120 pcf (1.92 g/cm³), ϕ = 30 degrees, c = 0, x_0 = 5 ft (1.52 m), gives z = 0.12475, y_1 = 2.495 ft (76.05 cm), and h = 7.484 ft (228.14 cm). Substituting in Eq. 27a gives E = 6,777 plf (10.1 g/m). If a line is taken from (0, y_0) to (x_0, 0), using the Coulomb wedge, it gives E = 6,530 plf (9.73 g/m), a 3.8% difference over the derived. Now if a Coulomb method is used with y_3 as shown in Fig. 8, the wedge can be taken as the Coulomb line with a uniform surcharge γy_3 . This gives E = 5,428 plf (8.09 g/m), where the overburden y_3 = 11.34 ft (3.46 m). Thus, the Coulomb wedge does not give E_{max} and a difference of 25% is obtained over the derived.

For condition (**ii**), where the slip surface must pass through a line $y = y_1$, $\Re$ in Eq. 7 can be rewritten as

$$\Re = -\gamma\frac{1+x'\tan\phi}{\tan\phi - x'}yx' - c\left[x' - \frac{1+x'\tan\phi}{\tan\phi - x'}\right] \qquad (40)$$

If the Euler Eq. is used with Eq. 40, the same slip surface will be obtained. Thus, for a slip surface passing through a line $y = y_1$, it yields

$$\left.\frac{\partial \Re}{\partial x'}\right|_{y=y_1} = 0 \qquad (41)$$

Executing Eq. 41 on Eq. 40 yields

$$\gamma y_1 \frac{\tan\phi\left(x'^2 - 2x'\tan\phi - 1\right)}{(\tan\phi - x')^2} - \frac{c\left(x'^2 - 2x'\tan\phi - 1\right)}{(\tan\phi - x')^2} = 0 \qquad (42)$$

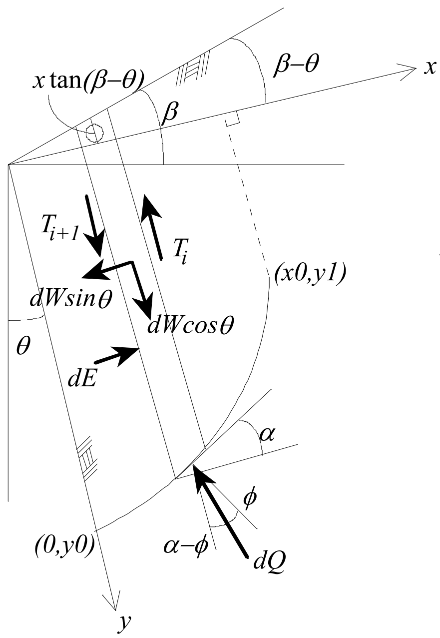
Thus,

$$\left. x' \right|_{y=y_1} = \tan\phi \pm \sqrt{\tan^2 + 1} = \tan\phi \pm \frac{1}{\cos\phi} \qquad (43)$$

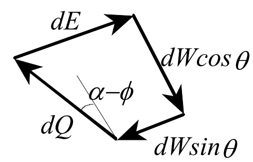
This forces *h* in Eq. 16 to be zero, or the Coulomb wedge is the solution for this condition. This situation is similar to having a uniform surcharge at the line $y = y_1$. This confirms that the Coulomb wedge is the proper slip surface as described in Eq. 25 and in the extremum of Eq. 26.

Analysis 2

For a sloped smooth wall with a sloped back fill with $c = 0$ (see Fig. 9(a)) the slip surface



Proposed Slip Surface
FIG. 9(a)



Force Diagram
FIG. 9(b)

FIG. 9 - Slip Surface with a Sloped Backfill and Slanted Wall

for an active condition can be derived similarly. It is assumed that $T_i = T_{i+1}$ with $T_0 = 0$.

From the force diagram, Fig. 9(b), it yields

$$dE = dW\left[\cos\theta \tan(\alpha - \phi) + \sin\theta\right] \quad (44)$$

Thus,

$$E = \gamma \int_0^{x_0} \left[\cos\theta \tan(\alpha - \phi) + \sin\theta\right]\left[y + x\tan(\beta - \theta)\right] dx \quad (45)$$

where $dW = \left[y + x\tan(\beta - \theta)\right]dx$. Eq. 45 can be rewritten as

$$E = \gamma k_4 \int_0^{x_0} \left(\sin\theta - \frac{u' + \tan\zeta}{1 - u' \tan\phi} \cos\theta \right) u dx \quad \text{.......} (46)$$

where $k_4 = 1 + \tan\phi \tan(\beta - \theta)$, $\zeta = \phi - \beta + \theta$, $u = [y + x\tan(\beta - \theta)] / k_4$, and $u' = [y' + \tan(\beta - \theta)] / k_4$. From variational method, and repeating the same analysis as in analysis 1, it yields

$$\frac{dx}{du} = \tan\phi - k_1 \sqrt{\frac{u}{u + h}} \quad \text{.......} (47)$$

and the slip surface

$$x = u\tan\phi - k_1 \left(\sqrt{u^2 + hu} - \frac{h}{2} \ln\left| 2\sqrt{u^2 + hu} + 2u + h \right| \right) + k \quad \text{.......} (48)$$

where $k_1 = \sqrt{[(1 + \tan\phi \tan\zeta)\tan\phi] / (\tan\zeta - \tan\theta)}$, and k is the constant of integration. Substituting Eq. 47 in Eq. 46 and rewriting yields

$$E = \gamma k_4 \int_{u_1}^{u_0} \left(k_3 - k_2 \frac{2u + h}{\sqrt{u^2 + hu}} \right) u du \quad \text{.......} (49)$$

where

$u_0 = y_0 / k_4$,

$u_1 = [y_1 + x_0 \tan(\beta - \theta)] / k_4$,

$k_2 = \cos\theta\sqrt{\tan\phi(\tan\zeta\tan\phi+1)(\tan\zeta-\tan\theta)}$, and

$k_3 = -\sin\theta\tan\phi + \cos\theta(1+2\tan\zeta\tan\phi)$.

Note E is maximum at $h = 0$, giving the slip surface to be the Coulomb wedge. Thus the application of β and θ is the same as in Coulomb. Validation of the solution at $h = 0$ can be checked numerically against US Army Corps Publication EM 1110-2-2502 Equation 3-13 and 3-19 as a different mathematical form of the same solution. Terzaghi (1943) [12], Rosenfarb and Chen (1972) [9] show for high β and $\theta = 0$ on a smooth wall, the slip surface is not a line and produces lower K_p values. However, for K_a values the differences were negligible. This means that the T_i's due to the ramp are not zero in passive condition. This is possible for a high $\beta > 0$ influencing the slice contact shear at the ramp. Equations derived for these conditions are shown in analyses 4 and 5 below. Evaluating E in Eq. 49 yields

$$E = \gamma k_4 k_3\left(\frac{u_0^2}{2}-\frac{u_1^2}{2}\right) - \gamma k_4 k_2\left[\left(u_0-\frac{h}{2}\right)\sqrt{u_0^2+hu_0} - \left(u_1-\frac{h}{2}\right)\sqrt{u_1^2+hu_1} + \frac{h^2}{4}\ln\left|\frac{2\sqrt{u_0^2+hu_0}+2u_0+h}{2\sqrt{u_1^2+hu_1}+2u_1+h}\right|\right] \quad (50)$$

For passive replace ϕ by $-\phi$ and E becomes:

$$E = \gamma k_4 k_3 \left(\frac{u_0^{\ 2}}{2} - \frac{u_1^{\ 2}}{2} \right) + \gamma k_4 k_2 \left[\left(u_0 - \frac{h}{2} \right) \sqrt{u_0^{\ 2} + hu_0} - \left(u_1 - \frac{h}{2} \right) \sqrt{u_1^{\ 2} + hu_1} + \frac{h^2}{4} \ln \left| \frac{2\sqrt{u_0^{\ 2} + hu_0} + 2u_0 + h}{2\sqrt{u_1^{\ 2} + hu_1} + 2u_1 + h} \right| \right] \qquad (51)$$

For condition (**i**) described before, the constant h and k can be obtained by substituting the two end points $(0, u_0)$ and (x_0, u_1) in the Eq. 48. For condition (**ii**): It is of merit to obtain the general solution, for the undetermined points, for a given curve and not just for a line. Some practical applications would be a buried vault, tunnel, tank, utility, etc. , that would interfere with the coulomb line. Suppose the given curve is $g_0(x, y) = 0$. By substituting for $y = k_4 u - x \tan(\beta - \theta)$ in $g_0(x, y)$, the new function will be $g(x, u) = 0$. If the end point (x_0, u_1) is obtained, then y_1 can be obtained from $y_1 = k_4 u_1 - x_0 \tan(\beta - \theta)$. From reference [15] the following condition must be satisfied for condition (**ii**):

$$\left. \frac{\partial \Re}{\partial u'} \right|_{x=x_0} - \frac{\left[\left. \frac{\partial g}{\partial u} \right|_{u=u_1} \right] \left[\left. \Re \right|_{x=x_0} \right]}{\left. \frac{\partial g}{\partial x} \right|_{x=x_0} + u_1' \left. \frac{\partial g}{\partial u} \right|_{u=u_1}} = 0 \qquad (52)$$

where $\Re$ is the integrand of Eq. 46. Doing the mathematics of Eq. 52, and solving for u_1' and using a relation for h from Eq. 47, yields

$$h = \left[\frac{1 - R(u_1)k_1^{\,2} \cot\phi}{-R(u_1)k_1 + \sqrt{1 - \dfrac{R(u_1)(1+\tan\phi\tan\theta)}{(\tan\zeta - \tan\theta)}}} \right]^2 u_1 - u_1 \quad \text{...} \quad (53)$$

where

$$R(u_1) = \frac{\left.\dfrac{\partial g(x,u)}{\partial x}\right|_{x=x_0}}{\left.\dfrac{\partial g(x,u)}{\partial u}\right|_{u=u_1}} \quad \text{..} \quad (54)$$

$R(u_1)$ is a function of u_1 alone is because x_0 can be determined from $g(x_0, u_1) = 0$. Now, k in Eq. 48 is determined from the point $(0, u_0)$. Thus, substituting Eq. 52 in Eq. 48 and replacing (x, y) by (x_0, u_1) gives an equation with u_1 as the only unknown. Hence, u_1 can be determined (numerically).

The following are two important functions:

A line: $y = ax + b \xrightarrow{\text{change to}} k_4 u - \left[\tan(\beta - \theta) + a\right]x - b = 0 = g(x,u)$

$$\xrightarrow{\text{from Eq 63}} R(u_1) = -\frac{\tan(\beta - \theta) + a}{k_4} \quad \text{..} \quad (55)$$

A circle: $(x - x_c)^2 + (y - y_c)^2 = r^2 \xrightarrow{\text{to}} (x - x_c)^2 + (k_4 u - x\tan(\beta - \theta) - y_c)^2 - r^2 = 0$

$$\xrightarrow{\text{to}} R(u_1) = \frac{x_0 - x_c}{k_4\left[k_4 u_1 - x_0 \tan(\beta-\theta) - y_c\right]} - \frac{\tan(\beta-\theta)}{k_4} \quad \text{............... (56)}$$

where x_0 can be determined from the quadratic equation in $g(x_0, u_1) = 0$.

The following equations are the results for the condition where $k_4 = 0$:

Eq. 45 yields:

$$E = \gamma \int_0^{x_0} \left(\sin\theta + \cos\theta \cot\phi + \frac{k_5}{u'} \right) u\,dx \quad \text{..................... (57)}$$

$$\frac{dx}{du} = \frac{h}{u} - \frac{k_6}{2k_5} \quad \text{..................... (58)}$$

$$x = h \ln u - \frac{k_6}{2k_5} u + k \quad \text{..................... (59)}$$

$$E = \gamma \int_{u_0}^{u_1} \left(k_5 \frac{h^2}{u} - \frac{k_6{}^2}{4k_5} u \right) du \quad \text{..................... (60)}$$

$$E = \gamma \frac{k_6{}^2}{8k_5} \left(u_0{}^2 - u_1{}^2 \right) - \gamma k_5 h^2 \ln\left| \frac{u_0}{u_1} \right| \quad \text{..................... (61)}$$

where $u = y + x\tan(\beta - \theta)$, $u' = y' + \tan(\beta-\theta)$, $u_0 = y_0$, $u_1 = y_1 + x_0 \tan(\beta - \theta)$, $k_5 = \cos\theta / \sin^2\phi$, $k_6 = \cos(\theta - \phi) / \sin\phi$, and k is the constant of integration. For passive replace ϕ by $-\phi$. For Condition (**ii**):

$$h = -\left[1 - \sqrt{1 - \frac{k_6 R(u_1)}{k_5}} \right]^2 \frac{u_1}{2R(u_1)} \quad \text{..................... (62)}$$

where $R(u_1)$ is of Eq. 54. To obtain the solution for a line or a circle, set $k_4 = 1$ in Eqs. 55 and 56, and u_1 can be determined numerically from Eq. 58.

For the passive condition replace ϕ by $-\phi$ in the above equations, starting with Eq. 44. It is noteworthy that if the axis in Fig. 9(a) is rotated by $-(\beta-\theta)$, the new axis will be $\bar{x} = -y\sin(\beta-\theta) + x\cos(\beta-\theta)$, $\bar{y} = y\cos(\beta-\theta) + x\sin(\beta-\theta)$, and $x = \bar{x}\cos(\beta-\theta) + \bar{y}\sin(\beta-\theta)$. If substituting for the new axis in the slip surface of Eq. 48, the resulting slip surface is similar to what has been obtained in Eq. 19, only with different constant values.

Analysis 3

When repeating analysis 2 with cohesion the Eq. 44 become:

$$dE = [\cos\theta\tan(\alpha-\phi) + \sin\theta]dW - c[\cos\alpha + \sin\alpha\tan(\alpha-\phi)]ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (63)$$

Or

$$E = \int_{u_1}^{u_0}\left\{\gamma k_4\left[-\frac{\dot{u}+\tan\zeta}{1-\tan\phi\dot{u}}\cos\theta + \sin\theta\right]u - c\left[1 + \left(k_4\dot{u} - \tan(\beta-\theta)\right)\frac{\dot{u}+\tan\zeta}{1-\tan\phi\dot{u}}\right]\right\}dx$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (64)$$

And the variational equation becomes:

$$(a_1 u + a_4)\dot{u}^2 - 2(a_2 u + a_5)\dot{u} + a_3 u + a_6 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(65)}$$

Where:

$$\begin{aligned}
a_1 &= \gamma k_4 \cos\theta\left(\tan\theta \tan^2\phi + \tan\phi\right) \\
a_2 &= \gamma k_4 \cos\theta \tan\phi\left(\tan\theta - \tan\zeta\right) \\
a_3 &= \gamma k_4 \cos\theta\left(\tan\theta - \tan\zeta\right) \\
a_4 &- c\left(\tan^2\phi - \tan\zeta \tan\phi - \tan\theta \tan\phi \tan(\beta - \theta)\right) - h \\
a_5 &= -c\left(\tan\phi - \tan\zeta \tan\phi \tan(\beta - \theta)\right) - h \tan\phi \\
a_6 &= -c - h
\end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(66)}$$

Thus the maximum E occurs at

$$\frac{dx}{du} = \frac{a_2 u + a_5}{a_3 u + a_6} - \sqrt{\left(\frac{a_2 u + a_5}{a_3 u + a_6}\right)^2 - \frac{a_1 u + a_4}{a_3 u + a_6}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(67)}$$

This clearly Eq. 67 will not produce a line slip surface for c not equal zero and $h = 0$ also it can be shown that h cannot be zero for maximum or minimum E. Validation of the solution at $h = 0$ can be checked numerically against US Army Corps Publication EM 1110-2-2502 Equation 3-25 as a different mathematical form of the same solution.

Analysis 4

For a sloped smooth wall with a sloped back fill with $c = 0$ (as in Fig. 9(a)) the slip surface for a passive condition can be derived similarly for shear due to the ramp. It is assumed that

$$T_i = \frac{d}{dx}\left\{\frac{1}{2}K_0\left[\frac{\tan(\beta-\theta)x}{\cos\theta}\right]^2 \cos\theta\tan\phi\right\} = K_0\frac{\tan^2(\beta-\theta)}{\cos\theta}\tan\phi\, xdx \quad \text{……………. (68)}$$

Where the shear is assumed to occur from an average at rest pressure $K_0 = 1.06(1-\sin\phi)$ on the ramp subtracting from the weight dW. From the force yields

$$dE = dW\left[\cos\theta\tan(\alpha+\phi)+\sin\theta\right] \quad \text{……………………………………………………………… (69)}$$

Thus,

$$E = \gamma\int_0^{x_0}\left[\cos\theta\tan(\alpha+\phi)+\sin\theta\right]\left[y + Ax\right]dx \quad \text{……………………………………………… (70)}$$

where $dW = (y+Ax)dx = \left\{y + \left[\tan(\beta-\theta) - K_0\frac{\tan^2(\beta-\theta)}{\cos\theta}\tan\phi\right]x\right\}dx$. Where

$A = \tan(\beta-\theta) - K_0\dfrac{\tan^2(\beta-\theta)}{\cos\theta}\tan\phi$. Eq. 70 can be rewritten as

$$E = \gamma k_4\int_0^{x_0}\left(\sin\theta - \frac{u'+\tan\zeta}{1+u'\tan\phi}\cos\theta\right)udx \quad \text{……………………………………………………… (71)}$$

where $k_4 = 1 - A\tan\phi$, $\zeta = -\phi - \tan^{-1}\left[\tan(\beta-\theta) - K_0 \dfrac{\tan^2(\beta-\theta)}{\cos\theta}\tan\phi\right]$,

$u = [y + Ax]/k_4$, and $u' = [y' + A]/k_4$. From variational method, and repeating the same analysis as in analysis 1, it yields

$$\frac{dx}{du} = -\tan\phi - k_1\sqrt{\frac{u}{u+h}} \quad \text{.......... (72)}$$

and the slip surface

$$x = -u\tan\phi - k_1\left(\sqrt{u^2 + hu} - \frac{h}{2}\ln\left|2\sqrt{u^2+hu} + 2u + h\right|\right) + k \quad \text{.......... (73)}$$

where $k_1 = \sqrt{[(1 - \tan\phi\tan\zeta)\tan\phi]/(\tan\theta - \tan\zeta)}$, and k is the constant of integration. Substituting Eq. 72 in Eq. 71 and rewriting yields

$$E = \gamma k_4 \int_{u_1}^{u_0}\left(k_3 + k_2\frac{2u+h}{\sqrt{u^2+hu}}\right)u\,du \quad \text{.......... (74)}$$

where

$u_0 = y_0 / k_4$,

$u_1 = [y_1 + Ax_0]/k_4$,

$k_2 = \cos\theta\sqrt{\tan\phi(1 - \tan\zeta\tan\phi)(\tan\theta - \tan\zeta)}$, and

$k_3 = \sin\theta\tan\phi + \cos\theta(1 - 2\tan\zeta\tan\phi)$.

For E Eq. 51 applies.

Note: E is minimum at $h = 0$, giving the slip surface to be a line. Note: as the slip surface approach the top surface the shear of Eq. 68 at $y < 0$ can be reduced. If $h = 0$ is used in Eq. 51 it yields:

$$E = \gamma \frac{k_3}{k_4} \frac{{y_0}^2}{2} + \gamma \frac{k_2}{k_4} {y_0}^2 \quad \text{...} \quad (75)$$

In here $u_0 = \dfrac{y_0}{k_4}$ and $u_1 = 0$, comparing with log-spiral in Table 1 at $\phi = \beta > 0$ and $\theta = 0$ yields:

Table 1 shows a maximum $\pm 5\%$ difference between log-spiral and Eq. 75.

TABLE 1 – Passive Pressure Coefficient K_p for Sloped Backfill $\phi = \beta > 0$ and $\theta = 0$

ϕ (degrees)	K_p (Coluomb)	K_p (log-spiral)	K_p (Eq. 75)
10	1.704	1.642	1.697
15	2.321	2.170	2.284
20	3.312	3.119	3.172
25	5.074	4.822	4.600
30	8.743	7.472	7.107
35	18.82	12.67	12.14
40	70.92	23.58	24.84

Analysis 5

Analysis 5 can be done similar to analysis 4. Where the ramp shear is

$$T_i = \frac{d}{dx}\left\{\frac{1}{2}K_0\left[\frac{\tan(\beta-\theta)x}{\cos\theta}\right]^2 \cos\theta\tan\phi + c\tan(\beta-\theta)x\right\} \text{ .... (76)}$$

Or

$$T_i = -K_0\frac{\tan^2(\beta-\theta)}{\cos\theta}\tan\phi\, xdx - c\tan(\beta-\theta)dx \text{ .... Passive .... (77)}$$

Surcharge

If there is a uniform surcharge q on top of the wall, it can be replaced by soil such that the soil thickness above y is $y_s = q/\gamma$. Thus y becomes $y + y_s$ in the integral equations. By making a change of variable $y_t = y + y_s$, the results will be similar and the equations easily modifiable.

Wall Pressure

The location of the Coulomb force is at $(2/3)y_0$ from the top of wall for triangular pressure. This is a reasonable assumption since K_a is independent of y_0, yielding $dE/dy_0 = \gamma K_a y_0$. However, this method is not applicable for examples 1, 2 and 3. A different approach is necessary in order to find the pressure. This can be done by moving down the wall from the top at incremental distances $y + \Delta y$, and find the potential slip surfaces, and forces. Thus, a table of E_j and y_j can be created. The stresses at a distance y_j can be taken as $(E_j - E_{j-1}) / (y_j - y_{j-1})$. This will give the same result as in a Coulomb condition. For example 1, 2, and 3, this method will take into account the Coulomb conditions at the top of the wall. One needs to examine the wall boundaries and movements before using the method. The following considerations need to be examined: (1) The potential slip surfaces above y_0 assumes the friction is almost fully mobilized. (2) The friction on the wall may vary. It may not be constant throughout. (3) No abrupt changes in deflection (the wall is continuous). In assumption (1), the friction on the bottom of the slice is $\phi' \leq \phi$. However, if the wall movements indicate the entire wedge is either active or passive, then $\phi' \cong \phi$. $\phi' \neq \phi$ because the soil above and below y_j are moving together resulting in the apparent slip surface on the bottom of the wall. However, they must be close. In consideration (2), even on a Coulomb wedge, if the friction of the wall varies it will produce non-linear pressure.

Irregular Backfill

If the wall does not have a level or sloped backfill but an irregular backfill, the analysis will require a computer program algorithm. One possibility is to use Fourier Series converted to a Taylor Series on the top surface and proceed to solve the Euler equation.

Conclusion

Variational method has been used to determine active and passive forces for a smooth wall with a cohesive and cohesionless soil. The methods are classical, conventional, and only practical assumptions were used. The resulting slip surface shows that the extremum of the force for a level backfill occurs when the slip surface is the Coulomb line for cohesion and cohesionless soil. Additionally, in order to have the Coulomb failure surface, the internal shear in the Bishop slices is required to be zero. For a sloped backfill the resulting slip surface shows that the extremum of the force occurs when the slip surface is a curve for cohesion soil and also when the internal shear in the Bishop slices is not zero. When adding the internal shear due to a sloped backfill for a cohesiveless material on a vertical wall the passive pressure is in good agreement with log spiral method; however, the slip surface is still a line.

For special cases where the slip surface is dictated by physical conditions and must pass through a point, a line or a curve, the forces and the slip surface can be obtained for both active and passive conditions. Also, a method of calculating the pressures on the wall is given. It can be seen that this method is adequate and will always give the derived slip surface. It has been noted in the examples given that the derived slip surface has not produced a significant difference over using a straight line. Although these differences are of desirable accuracies, having the correct slip surface is important when considering the influence of neighboring structures, activities, or discontinuities. If the engineer decides to use a conservative pressure by ignoring the neighboring structure and use a slip surface such that the neighboring structures do not exist, then the analysis given in this paper should guide the engineer on the differences in the pressures and give him the necessary comfort factor.

Appendix I

$$x'^2 - 2\tan\phi x' - \left(\frac{y - h_1 \tan^2\phi}{y + h_1} \right) = 0$$

Solving the quadratic equation yields

$$x' = \tan\phi \pm \sqrt{\tan^2\phi + \frac{y - h_1 \tan^2\phi}{y + h_1}}$$

or

$$x' = \tan\phi \pm \frac{1}{\cos\phi}\sqrt{\frac{y}{y + h_1}}$$

or

$$x' = \tan\phi \pm \frac{1}{\cos\phi}\sqrt{\frac{y + \dfrac{c}{\gamma\tan\phi}}{y + \dfrac{c}{\gamma\tan\phi} + h}}$$

Appendix II.-References

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Appendix III.- Notation

The following symbols are used in this paper:

α	= angle of the failure wedge, or of failure a slice, with the horizontal;
A	= constant;
A	= slope of line equation ($y = ax + b$);
β	= angle from the horizontal for ramp of soil on top of wall;
B	= the y-axis intercept of line equation ($y = ax + b$);
C	= cohesion;
Δy	= incremental vertical distance;
E	= horizontal active force on the wall to maintain equilibrium;
E_i	= horizontal active force on a slice to maintain equilibrium;
E_j	= horizontal active force on wall at distance y_j;
ϕ	= angle of internal friction of soil;
ϕ	= immobile angle of friction soil;
$f(y)$	= mathematical function in calculating the weight of soil in the slip surface;
γ	= soil unit weight;
C	= cohesion
$g(x, u)$	= curve function where the slip surface must pass through in (x, u) coordinates;
$g_0(x, y)$	= curve function where the slip surface must pass through in (x, y) coordinates;
H	= mathematical coefficient in the slip surface equations related to h_0;
h_0	= mathematical coefficient in the slip surface equations;

I = integer counter;

J = integer counter;

K_0 = at rest earth pressure coefficient;

K = constant of integration;

k_1 to k_6 = mathematical coefficient for representations slip surface and active force Eq.'s;

K_a = active earth pressure coefficient;

K_p = passive earth pressure coefficient;

λ = dimensionless coefficient;

N = integer counter;

P_a = horizontal active force on a tieback wall to maintain equilibrium;

p_a = horizontal active force above center of tieback grout length;

Q = reactive force on bottom of failure wedge or slice to maintain equilibrium;

Q = uniform surcharge pressure on top of wall;

$q(y)$ = mathematical horizontal pressure function;

$\Re$ = calculus of variation function of mixed variables representing the integrand;

$R(u_1)$ = a non dimensional function = $(\partial g / \partial x) / (\partial g / \partial u)$ at $u = u_1$ and $x = x_0$;

R = the radius of a circle;

θ = angle from the vertical for slant wall;

T = $T_{\max}$;

T_{design} = design tieback tension force;

T_i = vertical shearing force on a slice to maintain equilibrium;

$T_{\max}$ = maximum tieback tension force to failure;

U = new variable of distance as function of x and y;

u_0 = y_0;

u_1 = new variable of distance as function of x_0 and y_1;

W = vertical force from soil weight;

ξ = tieback angle from horizontal;

X = coordinate x-axis;

$\bar{x}$ = rotated coordinate of x-axis;

x_0 = x-coordinate at the end of the slip surface on top of the wall or in the soil;

x_c = the x-coordinate of the center of a circle;

ψ = directional angle from the vertical for the reactive force Q;

Y = coordinate y-axis;

$\bar{y}$ = rotated coordinate of y-axis;

y_0 = height of wall;

y_1 = y-coordinate point on the end of the slip surface in the soil;

y_2 = distance from top of the wall to tieback at face of the wall;

y_3 = distance from top of the wall to Coulomb wedge;

y_c = the y-coordinate of the center of a circle;

y_j = the $y + \Delta y$ incremental distance on the wall for active force E_j;

y_s = equivalent soil height to produce surcharge pressure q;

y_t = $y + y_s$;

ζ = $\phi - \beta + \theta$; and

Z = dimensionless coefficient.