Exact and Numerical Solutions for Large Deflection of Elastic Non-Prismatic Beams

by

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Abstract:

The solution for large deflections of beams that has not been solved in general in 260 years is now presented in this paper for point loads and moments in any directions along the beam with various end conditions. Also, the solution can be used for non-prismatic beams with various end conditions and numerical solution is presented to obtain exact solutions. Curvilinear beams and extensibility along the beam are also addressed.

Introduction:

The large deflection of beams has been investigated by Bisshopp and Drucker [1] for a point load on a cantilever beam. Timoshenko and Gere [2] developed the solution for axial load. Virginia Rohde [3] developed the solution for uniform load on cantilever beam. John H. Law [4] solved it for a point load at the tip of the beam and a uniform load combined. In this paper the general solution developed for a prismatic beam and in some cases for non-prismatic. However, numerical integration maybe needed along with solving compatibilities equations for the constants of integrations. A more general and preferable numerical solutions for a non-prismatic beam is also given using only many point loads acting with an angle on the beam with a moment on the node representing the approximate load. This point load can take any direction on the

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beam bending in the *x-x* direction or *y-y* direction of the moment of inertia. Thus the load is to be resolved to *x-x* direction and *y-y* direction of the moment of inertia in each orthogonal deflection given two non-linear differential equations. By solving each non-linear differential equation the orthogonal deflection components can be obtained.

An approximation attempt has been investigated by Scott and Caver [5] for all problems in which the moment can be expressed as a function of the independent variable. However, the solution presented here is not an approximation and is of a closed form. As a consequence one can take a load function and divide it into line segments for a non-prismatic beam with no axial loads and by using the proposed solution the large deflection can be calculated with reasonable accuracy. Thus, by taking smaller and smaller increments the accuracy is improved.

The solution is based on solving the non-linear differential equation of Bernoulli-Euler beam theory;

$$\frac{\ddot{y}}{\left(\sqrt{1+(\dot{y})^2}\right)^3} = \frac{M(x)}{EI(x)} \qquad (1)$$

Where $M(x)$ is the moment in the direction that corresponds to the moment of inertia $I(x)$, E is the modulus of elasticity, y is one of the orthogonal deflection say for I_{x-x}. Thus, if the other deflection is desired, then another equation is needed where $M(x)$ is the moment in the direction that corresponds to the moment of inertia $I_{y-y}(x)$. It is assumed the modulus of elasticity is constant and the bending does not alter the length of the beam. Three different closed form solutions are investigated for three different cases of the non-linear differential equation. In many ways $M(x)$ is not known until the final deflection is known so will assume $M(x)$ is known in the equations.

Case I:

In this case assume $\dfrac{M(x)}{EI(x)} = f(x)$, a function of *x* only. Thus,

$$\frac{\ddot{y}}{\left(\sqrt{1+(\dot{y})^2}\right)^3} = f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Let $\dot{y} = \tan\theta \Rightarrow \sqrt{1+(\dot{y})^2} = \dfrac{1}{\cos\theta}$ and $\ddot{y} = \dfrac{\dot{\theta}}{\cos^2\theta}$

Where θ is a new variable, then substitute θ in Eq. 2 yields,

$$\dot{\theta}\cos\theta = f(x) \quad \Rightarrow \quad \int \cos\theta \; d\theta = \int f(x)\,dx + C_1 \quad \text{or} \quad \sin\theta = \int f(x)\,dx + C_1$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\left[1-\left(\int f(x)\,dx + C_1\right)^2\right]^{\frac{1}{2}}$$

Thus,

$$\dot{y} = \frac{\sin\theta}{\cos\theta} = \pm\frac{\int f(x)\,dx + C_1}{\sqrt{1-\left[\int f(x)\,dx + C_1\right]^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

$$y(x) = \pm\int \frac{\int f(x)\,dx + C_1}{\sqrt{1-\left[\int f(x)\,dx + C_1\right]^2}}\; dx + C_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}$$

Where, C_1 and C_2 are constants of integration. This off course the solution Scott and Carver approximated as an infinite series not realizing it can be expressed in a closed form. Note: Eq. 4 gives an integral solution where if the moment is approximated by a curve it can give a better approximation than small deflection equations approximations. It is seen that if the denominator of Eq. 4 is approximated as a unity it would give the standard solution for small deflection approximations. In the case of large deflection it is better to use point loads and moments for approximating the general loading because it will be shown that it can be presented as elliptical integrals of the first and second kind. Elliptical solutions enable us to have a closed form solution as it has been successfully done through recursion [7, 8] instead of integration.

Case II:

In this case assume $\frac{M(y)}{EI} = -g(y)$, a function of y only. This happens in buckling problems and the moment of inertia is considered constant. Thus,

$$\frac{\ddot{y}}{\left(\sqrt{1+(\dot{y})^2}\right)^3} = -g(y) \quad \text{……………………………………………………..……………} (5)$$

Let $\dot{y} = \frac{1}{\dot{x}} \quad \Rightarrow \quad \ddot{y} = -\frac{1}{\dot{x}^2}\frac{d\dot{x}}{dx} = -\frac{1}{\dot{x}^2}\frac{d\dot{x}}{dy}\frac{dy}{dx} = -\frac{\ddot{x}}{\dot{x}^3}$

By replacing yields,

$$\frac{\ddot{x}}{\left(\sqrt{1+(\dot{x})^2}\right)^3} = g(y) \quad \text{…………………………………………………..………………} (6)$$

By following case I analysis with interchanging x by y the solution becomes:

$$\dot{x} = \pm\frac{\int g(y)\,dy + C_1}{\sqrt{1-\left[\int g(y)\,dy + C_1\right]^2}} \quad \text{…………….…………………………………………..……...……} (7)$$

$$x(y) = \pm\int \frac{\int g(y)\,dy + C_1}{\sqrt{1-\left[\int g(y)\,dy + C_1\right]^2}}\,dy + C_2 \quad \text{…………………………………………...…………..} (8)$$

Case III:

In this case assume $\frac{M(x,y)}{EI} = h(ax+by)$, a function of *x* and *y*. This happens in combine bending and buckling problems with point loads and the moment of inertia is considered constant. Thus,

$$\frac{\ddot{y}}{\left(\sqrt{1+(\dot{y})^2}\right)^3} = h(ax+by) \quad \text{(9)}$$

Consider the rotation of axis and let:

$$u = \frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y$$
$$v = \frac{a}{\sqrt{a^2+b^2}}x - \frac{b}{\sqrt{a^2+b^2}}y \quad \text{(10)}$$

Where *u* and *v* are the new variables then $h(ax+by) = h\left(\sqrt{a^2+b^2}\,u\right)$

$$x = \frac{\sqrt{a^2+b^2}}{2a}(u+v) = A(u+v) \quad \text{where } A = \frac{\sqrt{a^2+b^2}}{2a}$$
$$y = \frac{\sqrt{a^2+b^2}}{2b}(u-v) = B(u-v) \quad \text{where } B = \frac{\sqrt{a^2+b^2}}{2b} \quad \text{(11)}$$

And,

$$\frac{dx}{du} = A(1+\dot{v})$$
$$\frac{dy}{du} = B(1-\dot{v}) \quad \text{(12)}$$

$$\frac{dy}{dx} = \frac{B}{A}\left[\frac{1-\dot{v}}{1+\dot{v}}\right] = \frac{a}{b}\left[\frac{1-\dot{v}}{1+\dot{v}}\right]$$

………….…………….....……. (13)

$$\frac{d^2 y}{dx^2} = \frac{a}{b}\frac{d}{du}\left[\frac{1-\dot{v}}{1+\dot{v}}\right]\frac{du}{dx} = \frac{a}{b}\frac{d}{du}\left[\frac{1-\dot{v}}{1+\dot{v}}\right]\frac{1}{A(1+\dot{v})} = \frac{a}{Ab}\frac{-2\ddot{v}}{(1+\dot{v})^3}$$

Or

$$\frac{\ddot{y}}{\left(\sqrt{1+(\dot{y})^2}\right)^3} = \frac{-2a}{Ab}\frac{\ddot{v}}{(1+\dot{v})^3}\left[\frac{1}{1+\frac{a^2}{b^2}\left(\frac{1-\dot{v}}{1+\dot{v}}\right)^2}\right]^{\frac{3}{2}} = -\frac{4a^2b^2}{\left(a^2+b^2\right)^2}\frac{\ddot{v}}{\left[\left(\dot{v}+\frac{b^2-a^2}{b^2+a^2}\right)^2+1-\left(\frac{b^2-a^2}{b^2+a^2}\right)^2\right]^{\frac{3}{2}}}$$

…………………….….…………. (14)

Let

$$\dot{v}+\frac{b^2-a^2}{a^2+b^2} = \sqrt{1-\left(\frac{b^2-a^2}{a^2+b^2}\right)^2}\tan\theta$$

………………………………………………….…….. (15)

$$\ddot{v} = \sqrt{1-\left(\frac{b^2-a^2}{a^2+b^2}\right)^2}\frac{\dot{\theta}}{\cos^2\theta}$$

Where θ is a new variable, then substitute θ in Eq. 14 yields;

$$\dot{\theta}\cos\theta = -h\left(\sqrt{a^2+b^2}u\right) \quad\Rightarrow\quad \sin\theta = -\int h\left(\sqrt{a^2+b^2}u\right)du + C_1$$

…………………....………….. (16)

$$\cos\theta = \pm\sqrt{1-\left[-\int h\left(\sqrt{a^2+b^2}u\right)du + C_1\right]^2}$$

$$\dot{v} = \sqrt{1-\left(\frac{b^2-a^2}{a^2+b^2}\right)^2}\tan\theta - \frac{b^2-a^2}{a^2+b^2}$$

$$= \mp\sqrt{1-\left(\frac{a^2-b^2}{a^2+b^2}\right)^2}\frac{\int h\left(\sqrt{a^2+b^2}u\right)du\ +C_1}{\sqrt{1-\left[\int h\left(\sqrt{a^2+b^2}u\right)du\ +C_1\right]^2}} + \frac{a^2-b^2}{a^2+b^2}$$

……………………………..……………. (17)

$$v = \mp\sqrt{1-\left(\frac{a^2-b^2}{a^2+b^2}\right)^2}\int\frac{\int h\left(\sqrt{a^2+b^2}u\right)du\ +C_1}{\sqrt{1-\left[\int h\left(\sqrt{a^2+b^2}u\right)du\ +C_1\right]^2}}du + \frac{a^2-b^2}{a^2+b^2}u + C_2 \quad \text{………………..... (18)}$$

Thus, after integrating with respect to *u*, substitute *x* and *y* from Eq. 10 and an explicit equation of deflection in *x* and *y* is obtained. Another way of calculating *x* and *y* is pick *u* find *v* from Eq. 18 then find *x* and *y* from Eq. 11.

Hence, the solution for the nonlinear differential equation has been obtained for three cases and applications will follow.

<u>Application for Case I - Numerical Solution for Any Load Function Non-Prismatic Beam</u>:

This example is to demonstrate the solution for a cantilever beam. Other boundary conditions for beams are similar. First divide the beam into segmental beams of each length L_i and on each node of the segment insert the equivalent load P_i and moment Q_i to approximate the real load, see FIG.1. The moment on the segment beam at x_i is:

$$M_0 = P_0(x - x_0) + Q_0 \quad \text{for} \quad x_0 \le x < x_1$$

$$M_1 = P_0(x - x_0) + Q_0 + P_1(x - x_1) + Q_1 \quad \text{for} \quad x_1 \le x < x_2$$

.

.

$$M_i = \sum_{j=0}^{i} P_j(x - x_j) + Q_j \quad \text{for} \quad x_i \le x < x_{i+1} \qquad \ldots\ldots\ldots\ldots (19)$$

.

.

$$M_{n-1} = \sum_{j=0}^{n-1} P_j(x - x_j) + Q_j \quad \text{for} \quad x_{n-1} \le x < x_n$$

Where, all x_i are unknown.

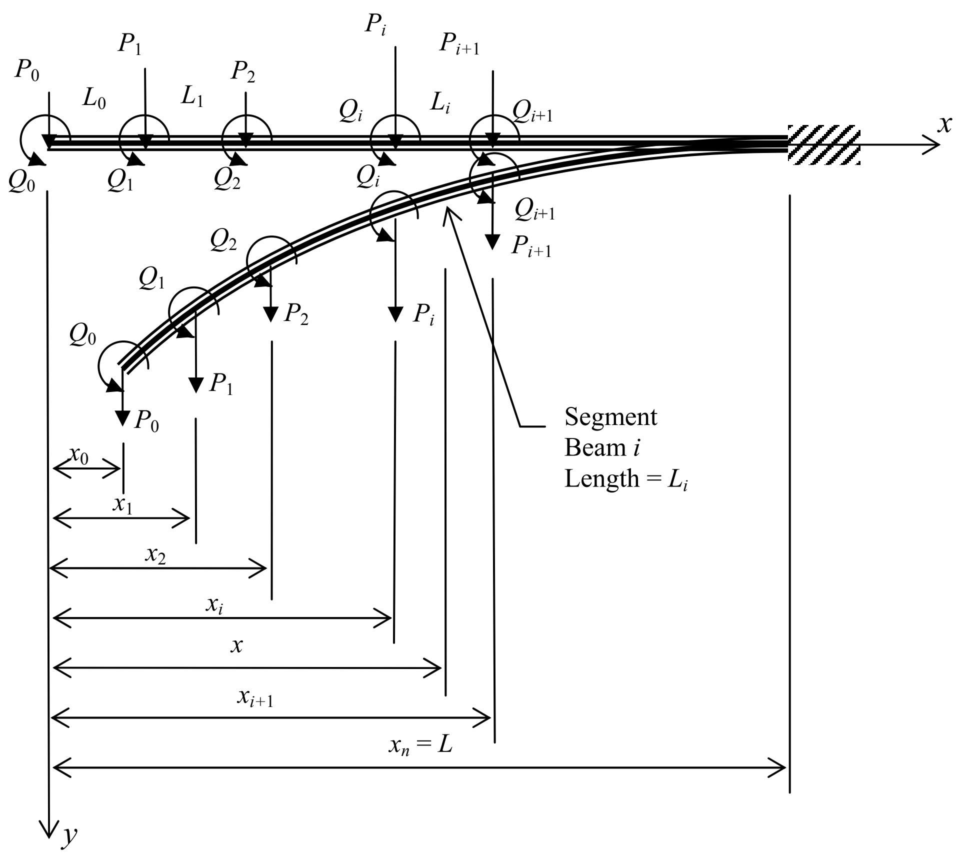

FIG.1 – Cantilever Beam Analysis – Non-Prismatic Beam

Point Loads are in the *y* direction

Now the moment of inertia:

$$I_i = I(L_{Ti}) \quad \text{where} \quad L_{Ti} = \sum_{j=0}^{i} L_i \qquad \text{.......... (20)}$$

Where, the moment of inertia is approximated at each beam segment to be constant[2] and the moment of inertia function is assumed continuous. Thus:

$$f_i(x) = \frac{M_i}{EI_i} = \frac{1}{EI_i}\left[\sum_{j=0}^{i} P_j(x - x_j) + Q_j\right] \qquad \text{.......... (21)}$$

Substitute Eq. 21 in Eq. 3 and find the slope on the segmental beam *i* yield:

$$\dot{y}_i(x) = \frac{\dfrac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x - x_j)^2 + Q_j(x - x_j)\right] + C1_i}{\sqrt{1 - \left[\dfrac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x - x_j)^2 + Q_j(x - x_j)\right] + C1_i\right]^2}} \quad \text{for} \quad x_i \le x \le x_{i+1}$$

.... (22)

Apply compatibilities yield:

$$\underset{\text{Seg. Bm. } i-1 \text{ to Seg. Bm. } i}{\dot{y}_{i-1}(x_i) = \dot{y}_i(x_i)} \quad \text{at} \quad x = x_i \qquad \text{.......... (23)}$$

At $x = L$, where L is the length of the beam at $x = x_n = L$,

$$\dot{y}_n(x_n) = \dot{y}_n(L) = 0 \qquad \text{.......... (24)}$$

Where n is the number of beam segments and $n+1$ is the total numbers of beam segments; apply Eq. 24 in Eq. 22 yields:

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[2] Note: this approximation does not mean there are stress singularities due to sharp corners at the discontinuities where each segment meets. All it means is the actual deflection of that segment can be approximated with the deflection of a beam with constant moment of inertia.

$$\dot{y}_n(L)=0 \quad \Rightarrow \quad \frac{1}{EI_{n-1}}\left[\sum_{j=0}^{n-1} 0.5P_j(L-x_j)^2+Q_j(L-x_j)\right]+C1_{n-1}=0$$

or ……..……….. (25)

$$C1_{n-1}=C1=-\frac{1}{EI_{n-1}}\left[\sum_{j=0}^{n-1} 0.5P_j(L-x_j)^2+Q_j(L-x_j)\right] \quad \text{since} \quad P_n=Q_n=0$$

When applying Eq. 23 for all i yield:

$C1_0=C1_1=C1_2=\cdots\cdots=C1_{n-1}=C1$ and

$$\dot{y}_i(x)=\frac{\dfrac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x-x_j)^2+Q_j(x-x_j)\right]+C1}{\sqrt{1-\left[\dfrac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x-x_j)^2+Q_j(x-x_j)\right]+C1\right]^2}} \quad \text{for} \quad x_i\le x\le x_{i+1}$$

…………………………….………… (26)

assuming $I_{i-1}=I_i$ at the joints (See Appendix B). Now impose the length of the beam segment to be un-extendible, yields,

$$L_i=\int_{x_i}^{x_{i+1}}\sqrt{1+\left(\dot{y}_i(x)\right)^2}\,dx \qquad \text{or}$$

$$L_i=\int_{x_i}^{x_{i+1}}\frac{dx}{\sqrt{1-\left[\dfrac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x-x_j)^2+Q_j(x-x_j)\right]+C1\right]^2}}$$ ………………....…….………. (27)

Eq. 27 does not lend itself to a simple solution (see appendix D for setting up Elliptic functions) and numerically complex. To simplify the equation assume the increments are small enough such that the slope throughout the interval of $x_i \le x \le x_{i+1}$ is the same (see Appendix A for the general solution), so:

$\dot{y}_i(x_i)\cong\dot{y}_i(x_{i+1})$ ……………………………………………...………………….. (28)

Thus,

$$L_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \Delta x_i \sqrt{1 + \left(\dot{y}_i(x_i)\right)^2} \qquad \text{(29)}$$

Or

$$L_i = \frac{x_{i+1} - x_i}{\sqrt{1 - \left[\frac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x_i - x_j)^2 + Q_j(x_i - x_j)\right] + C1\right]^2}} \qquad \text{(30)}$$

Thus, at $i = 0$ yields:

$$L_0 = \frac{x_1 - x_0}{\sqrt{1 - \left[\frac{1}{EI_0}\left[0.5P_0(x_0 - x_0)^2 + Q_0(x_0 - x_0)\right] + C1\right]^2}} = \frac{x_1 - x_0}{\sqrt{1 - [C1]^2}} \qquad \text{(31)}$$

At $i = 1$ yields

$$L_1 = \frac{x_2 - x_1}{\sqrt{1 - \left[\frac{1}{EI_1}\left[0.5P_0(x_0 - x_1)^2 + Q_0(x_0 - x_1)\right] + \frac{1}{EI_1}\left[0.5P_1(x_1 - x_1)^2 + Q_1(x_1 - x_1)\right] + C1\right]^2}}$$

$$= \frac{x_2 - x_1}{\sqrt{1 - \left[\frac{1}{EI_1}\left[0.5P_1(x_0 - x_1)^2 + Q_1(x_0 - x_1)\right] + C1\right]^2}} \qquad \text{(32)}$$

Substitute $x_1 - x_0$ from Eq. 31 in Eq. 32, for a given $C1$ and $x_2 - x_1$ is found.

At $i = 2$ yields

$$L_2 = \frac{x_3 - x_2}{\sqrt{1 - \left[\frac{1}{EI_2}\left[0.5P_0(x_2 - x_0)^2 + Q_0(x_1 - x_0)\right] + \frac{1}{EI_2}\left[0.5P_2(x_2 - x_1)^2 + Q_2(x_2 - x_1)\right] + C1\right]^2}}$$

$$\text{(33)}$$

Substitute $x_1 - x_0$ from Eq. 31 and $x_2 - x_1$ from Eq. 32 in Eq. 33, for a given $C1$ and $x_3 - x_2$ is found, where $x_2 - x_0 = (x_2 - x_1) + (x_1 - x_0)$.

Thus, if guessing $C1$ then find $x_{i+1} - x_i$, can be found since the denominator of Eq. 30 is always known from previous equations. And since $\sum_{i=0}^{n}(x_{i+1} - x_i) = x_n - x_0 = L - x_0$, x_0 can be found.

Therefore for a given $C1$ x_0 , x_1 , x_2 , ……. , x_{n-1} can be solved, then proceed by checking the end slope of Eq. 24 or Eq. 25. If it is not satisfied update $C1$ with numerical analysis until all the variables are found. For the deflection from Eq. 4 yields:

$$y_i(x) = \int \frac{\frac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x-x_j)^2 + Q_j(x-x_j)\right] + C1}{\sqrt{1 - \left[\frac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x-x_j)^2 + Q_j(x-x_j)\right] + C1\right]^2}}\,dx + C2_i \quad \text{for} \quad x_i \le x \le x_{i+1}$$

………………………...…………………. (34)

To find $C2_i$ assume compatibility and enforce:

$y_n(x_n) = y_n(L) = 0$ and find $C2_{n-1}$
$y_{n-1}(x_{n-1}) = y_n(x_{n-1})$ and find $C2_{n-2}$
etc.....

And the solution is found numerically.

<u>Application for Case II - Numerical Solution for Any Load Function Non-Prismatic Beam</u>:

This solution is very similar to the application of Case I or the previous application but instead of using x and x_i substitute for y and y_i and can find $C1$, y_0 , y_1 , …… , y_{n-1} and $C2_i$ for the deflection of Eq. 8 see FIG.2.

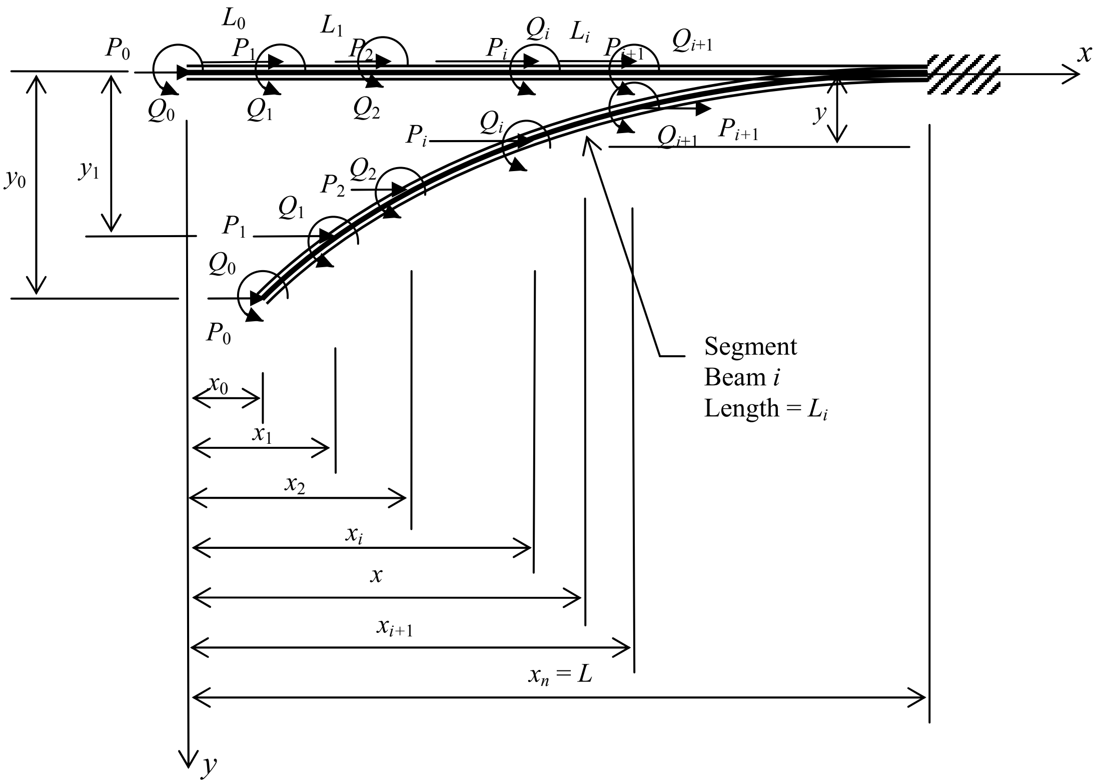

FIG.2 – Cantilever Beam Analysis – Non-Prismatic Beam

Point Loads are in the x direction

<u>Application for Case III - Numerical Solution for Any Load Function Non-Prismatic Beam</u>:

Again implement the solution using similar procedures as in the application of Case I, see FIG.3. If assume at x_i the resultant moment in the x and y direction is M_{xi} and M_{yi} and if assume at x_i the resultant force in the x and y direction is R_{xi} and R_{yi} , then the moment for Eq. 9 becomes:

$$h(ax+bx) = -(x-\bar{x}_i)\frac{R_{xi}}{EI_i} - (y-\bar{y}_i)\frac{R_{yi}}{EI_i} \quad \text{where}$$

$$\bar{x}_i = x_i + \frac{M_{xi}}{R_{xi}} \quad \text{and} \quad \bar{y}_i = y_i + \frac{M_{yi}}{R_{yi}} \qquad \ldots\ldots (34.1)$$

Thus, if translating temporarily the axis to a local axis with $\bar{x}_i$ and $\bar{y}_i$ Eq. 9 becomes:

$$\frac{\ddot{y}}{\left(\sqrt{1+(\dot{y})^2}\right)^3} = h(a_i x + b_i y) = a_i x + b_i y\,, \quad \text{where} \quad a_i = -\frac{R_{xi}}{EI_i} \text{ and } b_i = -\frac{R_{yi}}{EI_i} \quad \text{............ (34.2)}$$

Note: for segment *i*-1 and segment *i* at $x = x_i$ the resultants are the same so

$$a_i = a_{i-1} \quad \text{and} \quad b_i = b_{i-1} \quad \text{............ (34.3)}$$

Thus if impose

$$\dot{y}_{i-1}(x_i) = \dot{y}_i(x_i) \qquad \text{at} \qquad x = x_i$$
$$\text{Seg. Bm. } i-1 \quad \text{to} \quad \text{Seg. Bm. } i$$

Which means from Eq. 13 and 34.3 yields:

$$\dot{v}_{i-1}(u_i) = \dot{v}_i(u_i) \qquad \text{at} \qquad u = u_i \quad \text{............ (35)}$$

This implies $C1_0 = C1_1 = C1_2 = \cdots\cdots = C1_{n-1} = C1$, assuming $I_{i-1} = I_i$ at the joints (Se Appendix B). u_0 , u_1 , ……. , u_n can be found by using the approximation:

$$\dot{y}_i(x_i) \cong \dot{y}_i(x_{i+1}) \quad \text{or} \quad \dot{v}_i(u_i) \cong \dot{v}_i(u_{i+1}) \quad \text{............ (36)}$$

Thus,

$$L_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \Delta u_i \sqrt{\frac{\Delta x_i^2}{\Delta u_i^2} + \frac{\Delta y_i^2}{\Delta u_i^2}} = \Delta u_i \sqrt{\left(\frac{\sqrt{a_i^2 + b_i^2}}{2a_i}(1+\dot{v}_i)\right)^2 + \left(\frac{\sqrt{a_i^2 + b_i^2}}{2b_i}(1-\dot{v}_i)\right)^2}$$

$$\text{............ (37)}$$

Or

$$L_i \cong (u_{i+1} - u_i)\frac{\sqrt{a_i^2 + b_i^2}}{2}\sqrt{\left[\frac{1+\dot{v}_i(u_i)}{a_i}\right]^2 + \left[\frac{1-\dot{v}_i(u_i)}{b_i}\right]^2} \quad \text{............ (38)}$$

Which is again for a given $C1$ u_i's can be found from Eq. 38. When using Eq.10:

$\sum_{i=0}^{n-1}(u_{i+1} - u_i) = u_n - u_0 = \frac{a_{n-1}}{\sqrt{a_{n-1}^2 + b_{n-1}^2}} L - u_0$. Thus, all u_i's can be obtained by back substitutions with checking if $C1$ satisfies:

$$\dot{y}_n(x_n) = \dot{y}_n(L) = 0$$

or ……………………….……………… (39)

$$\frac{a_{n-1}}{b_{n-1}}\left(\frac{1-\dot{v}_i(u_n)}{1+\dot{v}_i(u_n)}\right) = \frac{a_{n-1}}{b_{n-1}}\left(\frac{1-\dot{v}_n\left(\frac{a_{n-1}}{\sqrt{a_{n-1}^2+b_{n-1}^2}}L\right)}{1+\dot{v}_n\left(\frac{a_{n-1}}{\sqrt{a_{n-1}^2+b_{n-1}^2}}L\right)}\right) = 0$$

Now, find $C2_{n-1}$ from the last segment to satisfy

$$v_n(u_n) = v_n\left(\frac{a_{n-1}}{\sqrt{a_{n-1}^2+b_{n-1}^2}}L\right) = \frac{a_{n-1}}{\sqrt{a_{n-1}^2+b_{n-1}^2}}L \quad \text{…………………………...………………….. (40)}$$

and find all of $C2_i$ from the compatibility equation of deflection y_i and using Eq. 11:

$u_{n-1} - v_{n-1}(u_{n-1}) = u_{n-1} - v_n(u_{n-1})$ and find $C2_{n-2}$

$u_{n-2} - v_{n-2}(u_{n-2}) = u_{n-2} - v_{n-1}(u_{n-2})$ and find $C2_{n-3}$

etc.....

When done find all of v_0 , v_1 , ……… , v_{n-1} then substitute in Eq. 11 to find x_i and y_i .

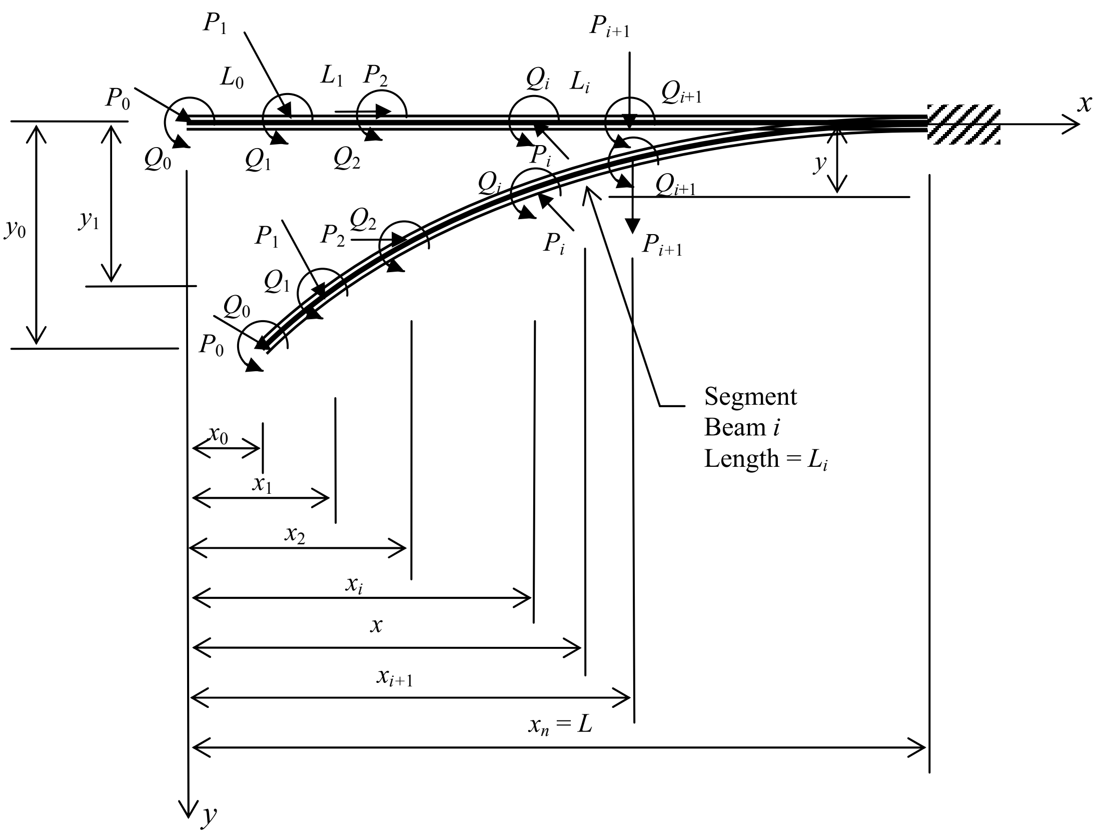

FIG.3 – Cantilever Beam Analysis – Non-Prismatic Beam

Point Loads are in any direction

<u>Application Examples of Case III Numerical Solution</u>:

(a) <u>Fishing Pole</u>: an application for Case III is shown on FIG. 4, where the beam has an angle α with the horizontal and a load P_0 and P_1 hanging from the beam. One common application is a fishing pole that has a fish that has the load P_0 from the vertical and P_1 = 0. In this case when varying the angle α with the horizontal the moment changes and the deflection curve change giving a smaller or bigger moment with various elastic curves. This affects the ability to pull the fishing with the real and having various controls on catching the fish by puling or letting go the line. An experienced fisherman does this procedure naturally and gets the credit for not loosing the fish. A good fishing rod would be designed to have a moderate elastic curve configuration when varying P_0 and the angle α. The equations for a non prismatic fishing pole can be:

$P_x = P_1 \cos\alpha$ and $P_y = P_1 \sin\alpha$ Thus the function h_i in Eq.9 becomes:

$$h_0(a_0 x, b_0 y) = \frac{P_0}{EI_0}\left[(x - x_0)\cos\alpha - (y - y_0)\sin\alpha\right]$$

or

$$h_i(a_i x, b_i y) = \sum_{j=0}^{i} \frac{P_j}{EI_i}\left[(x - x_j)\cos\alpha - (y - y_j)\sin\alpha\right] = a_i(x - \bar{x}_i) + b_i(y - \bar{y}_i)$$

where,

$$\bar{x}_i = \frac{\sum_{j=0}^{i} \frac{P_j x_j}{EI_i}}{\sum_{j=0}^{i} \frac{P_j}{EI_i}} \qquad \bar{y}_i = \frac{\sum_{j=0}^{i} \frac{P_j y_j}{EI_i}}{\sum_{j=0}^{i} \frac{P_j}{EI_i}} \qquad a_i = \sum_{j=0}^{i} \frac{P_j}{EI_i}\cos\alpha \quad \text{and} \quad b_i = -\sum_{j=0}^{i} \frac{P_j}{EI_i}\sin\alpha$$

In this situation each beam segment can be translated in a new local axis by $\bar{x}_i$ and $\bar{y}_i$ to have Eq. 10 ready for rotation of axis to satisfy Eq. 9 and then translate to the global axis. For example translate the local axis by $\bar{x}_i$ and $\bar{y}_i$, and substitute in Eq. 17 and 18 the new coordinates $(u - \bar{u}_i)$ and $(v - \bar{v}_i)$ for u and v in the solution of Eq. 17 and Eq. 18, where $\bar{u}_i$ and $\bar{v}_i$ are obtained from substituting $\bar{x}_i$ and $\bar{y}_i$ in equation 10 with the replacement of a by a_i and b by b_i.

(b) <u>Curved Beam</u>: Another application example where the beam is a non-prismatic curved beam. Thus, if subdivide the curved beam to a smaller segments cantilever straight beams[3] with constant moment of inertia as in using FIG. 3. If α_i is the angle the segmental beams make with the horizontal, then the problem can be solved by taking each deflection curve derived from the local axis of the segmental beam and rotated by the angle α_i then translate by $\bar{x}_i$ and $\bar{y}_i$ to the global axis and the problem can be solved numerically.

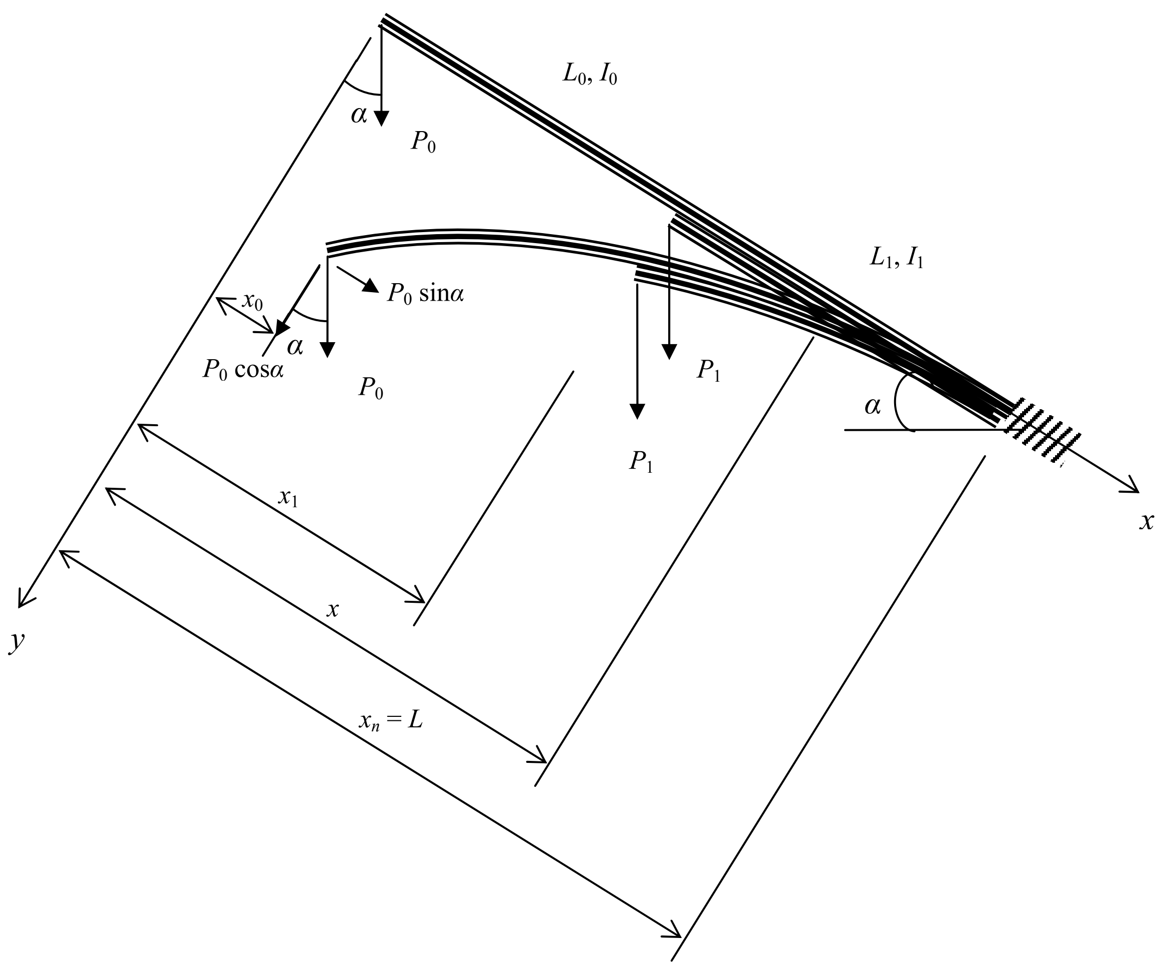

FIG.4 – Fishing Pole Example (a)

[3] Note: this approximation does not mean there are stress singularities due to sharp corners at the discontinuities where each segment meets. All it means is the actual deflection of the slightly curved segment can be approximated with the deflection of a beam with straight segment with a constant moment of inertia.

(c) Bow and Arrow: The ancient structural problem in archeries or shooting bow and arrow can finally be solved. Amazingly, can design a curved non-prismatic beam to give the proper deflection curve for a human precise measurement giving the best comfortable result for a more accurate bull's-eye. Possibly, a unique design for each athlete, so putting tension by changing the string size can be less desirable. To simplify the equations and show the example, a non-prismatic curved beam will not be used but use a non-prismatic straight beam, see FIG. 5. The function h_i in Eq. 9 becomes:

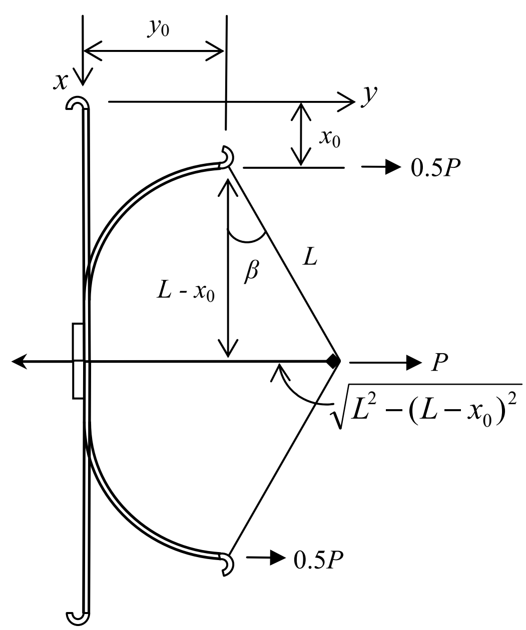

FIG.5 – Bow and Arrow Example (c)

$$h_0(a_0x, b_0y) = \frac{1}{EI_0}\left[(x-x_0)\left(\frac{P}{2}\right) - (y-y_0)\cot\beta\left(\frac{P}{2}\right)\right]$$

$$= \frac{1}{EI_0}\left[(x-x_0)\left(\frac{P}{2}\right) - (y-y_0)\frac{(L-x_0)}{\sqrt{L^2-(L-x_0)^2}}\left(\frac{P}{2}\right)\right]$$

$$h_i(a_i x, b_i y) = \frac{1}{EI_i}\left[(x-x_0)\left(\frac{P}{2}\right) - (y-y_0)\cot\beta\left(\frac{P}{2}\right)\right]$$

$$= \frac{1}{EI_i}\left[(x-x_0)\left(\frac{P}{2}\right) - (y-y_0)\frac{(L-x_0)}{\sqrt{L^2-(L-x_0)^2}}\left(\frac{P}{2}\right)\right]$$

$$= a_i(x-x_0) + b_i(y-y_0)$$

where

$$a_i = \frac{P}{2EI_i} \qquad \text{and} \qquad b_i = -\frac{P}{2EI_i}\frac{(L-x_0)}{\sqrt{L^2-(L-x_0)^2}}$$

In this situation each beam segment can be translated by x_0 and y_0 in its local axis to satisfy Eq. 9 and then to the global axis. For example translate by x_0 and y_0, and substitute in Eq. 17 and 18 by the new coordinates (u-u_i) and (v-v_i) for u and v of Eq. 17 and Eq. 18, where $\overline{u}_i$ and $\overline{v}_i$ are obtained from substituting x_0 and y_0 in equation 10 with the replacement of a by a_i and b by b_i.

<u>Column with Load through Fixed Point</u>: The problem of column with load through fixed point was presented by Timoshenko and Gere [2]. Jong-Dar Yau [6] presented a solution for Closed-Form Solution of Large Deflection for a Guyed Cantilever Column Pulled by an Inclination Cable. A more general problem is to allow the fixed point D to have a coordinate point (x_d, y_d) instead of the coordinate point (x_d, 0) as in FIG. 6. As if the tip of the column is attached by a cable with a shackle to point D and the shackle is being tightened. The column is assumed a non-prismatic. This situation can also happen in a vertical fishing pole, where the fish pulls with an angle β. Thus, moment becomes:

$$M = -(x - x_0)P_y - (y - y_0)P_x$$
$$= -(x - x_0)P\sin\beta - (y - y_0)P\cos\beta$$
$$h(a_i x, b_i y) = \frac{P}{EI_i}\left[(x - x_0)\frac{(y_d - y_0)}{\sqrt{(y_d - y_0)^2 + (x_d - x_0)^2}} - (y - y_0)\frac{(x_d - x_0)}{\sqrt{(y_d - y_0)^2 + (x_d - x_0)^2}}\right]$$
$$= a_i(x - x_0) + b_i(y - y_0)$$
where,
$$a_i = \frac{P}{EI_i}\frac{(y_d - y_0)}{\sqrt{(y_d - y_0)^2 + (x_d - x_0)^2}} \quad \text{and} \quad b_i = -\frac{P}{EI_i}\frac{(x_d - x_0)}{\sqrt{(y_d - y_0)^2 + (x_d - x_0)^2}}$$

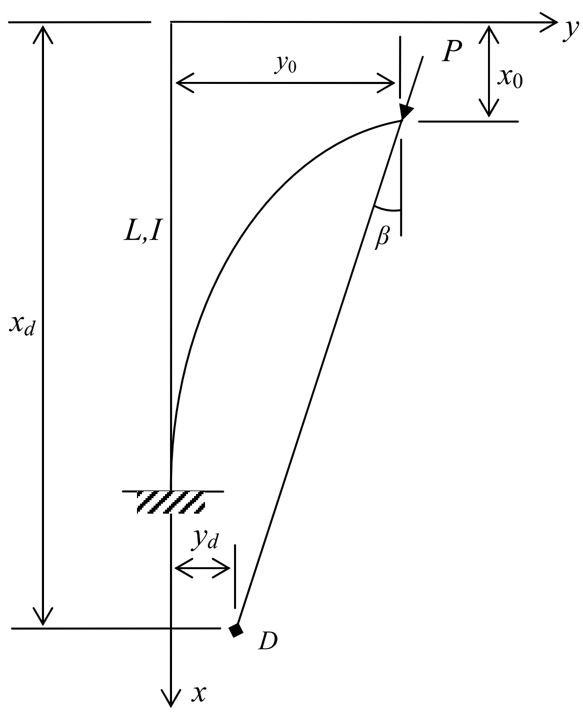

FIG.6 - Column with Load through Fixed Point (d)

In this situation each column segment can be translated by x_0 and y_0 in its local axis to satisfy Eq. 9 and then to the global axis. For example translate by x_0 and y_0, and substitute in Eq. 17 and 18 by the new coordinates $(u-u_i)$ and $(v-v_i)$ for u and v of Eq. 17 and Eq. 18, where $\bar{u}_i$ and $\bar{v}_i$ are obtained from substituting x_0 and y_0 in equation 10 with the replacement of a by a_i and b by b_i.

<u>The ± sign in Eq. 3, 7 and 17</u>:

The ± sign in the solution of Eq. 3, 7 and 17 can be used interchangeably when the slope of the deflection curve goes to infinity at a point and the slope change sign. Another word the deflection curve is not a function anymore and becomes circular. In Eq. 17 $\dot{v} \Rightarrow \infty$ when $\dot{y} \Rightarrow a/b$. At that point the correct sign of Eq. 17 must be used.

<u>Other End Condition</u>: See Appendix E

<u>Curvilinear Beams:</u>

For a curvilinear beam with a function $y = R(x)$ the new radius of curvature must satisfy the following equation:

$$r_{new} = \frac{1}{\dfrac{M(x)}{EI} + \dfrac{1}{r_{old}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(41)}$$

So that if $M(x) = 0$ the radius of curvature does not change and remain of the function $y = R(x)$.

Thus Eq. 1 becomes:

$$\frac{\ddot{y}}{\left(\sqrt{1+(\dot{y})^2}\right)^3} = \frac{M(x)}{EI(x)} + \frac{\ddot{R}(x)}{\left(\sqrt{1+\left[\dot{R}(x)\right]^2}\right)^3} = t(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(42)}$$

And solution is as Eq. 3 and Eq. 4 by replacing $f(x)$ by $t(x)$.

Extensibility:

In order to account for extensibility of the beam will analyze a beam segment L_i. Let α_i be the directional angle of the load P_i and θ_i the angle at P_i representing the slope of the beam at that point as in Fig. 7.

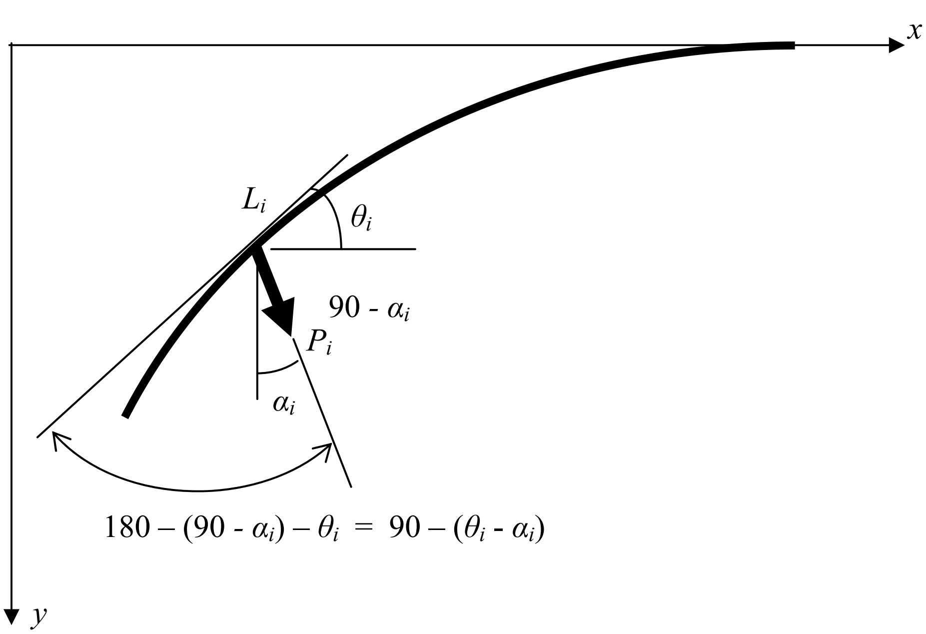

Fig. 7 Extensibility

For extensibility of a small arc length ds in L_i expressing the change in length (shortening) ε_i due to the axial load in L_i as:

$$\varepsilon_i = -\int_{x_i}^{x_{i+1}} \frac{P_i \sin(\theta - \alpha_i)}{A_i E} ds \quad \text{(43)}$$

Expressed as (ΣPL/AE) where A_i is the area of the segment L_i and the load P_i is the resultant at every ds. In a shortening condition, the shape of the deflection curve did not change only the curve has shrunk. The resultant moment is affect by extensibility due to the change of

x_i , y_i or u_i for a new $\bar{x}_i$, $\bar{y}_i$ or $\bar{u}_i$. Then for a given deflection curve derived without including extensibility that gives x_i , y_i or u_i yields:

$$\bar{x}_i = x_i + \sum_{j=0}^{i} \hat{\varepsilon}_j \hat{i} = x_i - \sum_{j=0}^{i} \int_{x_j}^{x_{j+1}} \frac{P_j \sin(\theta - \alpha_j)}{A_j E} \cos\theta ds$$

$$= x_i - \sum_{j=0}^{j} \int_{x_j}^{x_{j+1}} \frac{P_j \left(\sin\theta \cos\alpha_j - \cos\theta \sin\alpha_j\right)}{A_j E} \frac{dx}{ds} ds$$

$$= x_i - \sum_{j=0}^{j} \int_{x_j}^{x_{j+1}} \frac{P_j \left(\frac{dy}{ds} \cos\alpha_j - \frac{dx}{ds} \sin\alpha_j\right)}{A_j E} dx$$

$$= x_i - \sum_{j=0}^{j} \int_{x_j}^{x_{j+1}} \frac{P_j \left(\frac{dy}{dx} \cos\alpha_j - \sin\alpha_j\right)}{A_j E} \frac{dx}{ds} dx$$

$$= x_i - \sum_{j=0}^{j} \int_{x_j}^{x_{j+1}} \frac{P_j \left(\frac{dy}{dx} \cos\alpha_j - \sin\alpha_j\right)}{A_j E} \frac{1}{\sqrt{1 + (dy/dx)^2}} dx$$

$$= x_i - \sum_{j=0}^{j} \int_{x_j}^{x_{j+1}} \frac{P_j \left(\dot{y} \cos\alpha_j - \sin\alpha_j\right)}{A_j E} \frac{1}{\sqrt{1 + (\dot{y})^2}} dx \qquad \Leftarrow$$

$$\bar{y}_i = y_i + \sum_{j=0}^{i} \hat{\varepsilon}_j \hat{j} = y_i - \sum_{j=0}^{i} \int_{x_j}^{x_{j+1}} \frac{P_j \sin(\theta - \alpha_j)}{A_j E} \sin\theta ds \qquad \text{……………………. (44)}$$

$$= y_i - \sum_{j=0}^{j} \int_{x_j}^{x_{j+1}} \frac{P_j \left(\dot{y} \cos\alpha_j - \sin\alpha_j\right)}{A_j E} \frac{\dot{y}}{\sqrt{1 + (\dot{y})^2}} dx \qquad \Leftarrow$$

Alternatively, if would like to include the effect extensibility on the moments then Eq. 27 becomes[4]:

---

[4] Note the extensibility Eq. 45 is slightly conservative since $\int ds \neq L_i$ when integrate from x_i to x_{i+1}.

$$L_i - \varepsilon_i = \int_{x_i}^{x_{i+1}} \sqrt{1+\left[\dot{y}_i(x)\right]^2}\,dx$$

or

$$\begin{aligned}
L_i &= \int_{x_i}^{x_{i+1}} \sqrt{1+\left[\dot{y}_i(x)\right]^2}\,dx - \int_{x_i}^{x_{i+1}} \frac{P_i \sin(\theta-\alpha_i)}{A_i E} ds \\
&= \int_{\bar{x}_i}^{x_{i+1}} \sqrt{1+\left[\dot{y}_i(x)\right]^2}\,dx - \int_{x_i}^{x_{i+1}} \frac{P_i\left(\sin\theta\cos\alpha_i - \cos\theta\sin\alpha_i\right)}{A_i E}\sqrt{1+\left[\dot{y}_i(x)\right]^2}\,dx \\
&= \int_{x_i}^{x_{i+1}} \sqrt{1+\left[\dot{y}_i(x)\right]^2}\,dx - \int_{x_i}^{x_{i+1}} \frac{P_i\left(\frac{dy}{ds}\cos\alpha_i - \frac{dx}{dy}\sin\alpha_i\right)}{A_i E}\sqrt{1+\left[\dot{y}_i(x)\right]^2}\,dx \\
&= \int_{x_i}^{x_{i+1}} \sqrt{1+\left[\dot{y}_i(x)\right]^2}\,dx - \int_{x_i}^{x_{i+1}} \frac{P_i\left(\frac{\dot{y}_i(x)}{\sqrt{1+\left[\dot{y}_i(x)\right]^2}}\cos\alpha_i - \frac{1}{\sqrt{1+\left[\dot{y}_i(x)\right]^2}}\sin\alpha_i\right)}{A_i E}\sqrt{1+\left[\dot{y}_i(x)\right]^2}\,dx \\
&= \int_{x_i}^{x_{i+1}} \sqrt{1+\left[\dot{y}_i(x)\right]^2}\,dx - \int_{x_i}^{x_{i+1}} \frac{P_i\left[\dot{y}_i(x)\cos\alpha_i - \sin\alpha_i\right]}{A_i E}dx \\
&= \int_{x_i}^{x_{i+1}} \sqrt{1+\left[\dot{y}_i(x)\right]^2}\,dx + \frac{\left(y_{i+1}-y_i\right)P_i\cos\alpha_i}{A_i E} + \frac{\left(x_{i+1}-x_i\right)P_i\sin\alpha_i}{A_i E}
\end{aligned}$$

…………………………………….. (45)

This equation can be adjusted with the rotation of the axis for α_i of Case III and implemented per Eq. 13 and instead of updating x_i in Eq. 27 update u_i and Eq. 17 when substituting Eq. 17 in Eq. 13 to obtain $\dot{y}_i(x)$ of Eq. 45 to find x_i. This assumes the total deflection curve is shortened or elongated by $\varepsilon_x, \varepsilon_y$ and the solution in Eq. 4, 8, 18 remains the same and only is effected by x_i and y_i as in Eq. 19, where:

$$\begin{aligned}
\varepsilon_x &= -\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \frac{P_i \sin(\theta-\alpha_i)}{A_i E}\cos\theta ds \\
\varepsilon_y &= -\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \frac{P_i \sin(\theta-\alpha_i)}{A_i E}\sin\theta ds
\end{aligned}$$

…………………………………………………….. (46)

Finale note on extensibility and large deflections: extensibility may have a minor effect on the moments however it can affect the buckling deflection criterion. In general when loading a beam the moment and axial load reduces with time however the deflection increases with time until the final x_i , y_i or u_i occurring at t_{final} . In this case the safety factor on the stresses must account for the dynamic problem of loading and reloading and care must be taken when using large deflections in design.

Comparison with current methods for large deflections: It would be very difficult to draw conclusions from one or two examples when comparing the exact solutions with any approximate method including finite elements. Thus, comparison is left out to a more in-depth study in a different article. The finding in this paper stands alone on its own two feet, is complete and it is a bench mark.

Conclusion:

The closed form and general solution of non-linear differential equation of Bernoulli-Euler beam theory is solved numerically for general loading function for a non-prismatic beam and can be approximated for a non-prismatic curved beam when the presented solution of curvilinear beam is not used. In some cases it is solved in closed form for prismatic and non-prismatic beam. In general the Elastica, as called by Timoshenko and Gere [2], is solved.

References:

1. Bisshopp, K. E., and Drucker, D. C., "Large Deflections of Cantilever Beams," Quarterly of Applied Mathematics, Vol. 3, 1945, pp. 272-275
2. Timoshenko, S. P. and Gere, J. M., "Theory of Elastic Stability" 1961, McGraw-Hill Book Company, New York, pp. 76-82 and pp. 55-57
3. Rohde, F. V., "Large Deflection of Cantilever Beam with a Uniformly Distributed Load," Quarterly of Applied Mathematics, Vol. 11, 1953, pp. 337-338
4. Lau, John H. "Large Deflection of Cantilever Beams," Journal of Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 107, NO. EM1, February, 1981. pp. 259-264
5. Scott, E. J. and Carver, D. R. "On the Nonlinear Differential Equation for Beam Deflection," ASME Applied Mechanics Division June, 1955, pp. 245-248.
6. Jong-Dar Yau "Closed-Form Solution of Large Deflection for a Guyed Cantilever Column Pulled by an Inclination Cable" Journal of Marine and Technology, Vol. 18, No. 1, 2010, pp 120-136
7. Nellis, J. William "Tables of Elliptic Integrals" NASA Contractor Report NASA CR-289 Prepared under Grant No. NsG-293 by IOWA State University, Ames, Iowa for NATIONAL AERONAUTICS AND SPACE ADMINISTRATION . WASHINGTON, D. C. AUGUST 1965
8. B. C. Carlson "Three Improvements in Reduction and Computation of Elliptic Integrals" Journal of Research of National Institute of Standards and Technology, Volume 107, Number 5, September-October 2002, pp 413-418

APPENDIX A

General algorithm solution of solving for x_i for application example Case I:

From Eq. 27 let the function:

$$\Phi_i(x) = \int \frac{dx}{\sqrt{1-\left[\frac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x-x_j)^2 + Q_j(x-x_j)\right] + C1\right]^2}} + c \quad \ldots\ldots\ldots\ldots\ldots\ldots (47)$$

When setting $\frac{\partial \Phi_i}{\partial x} = 0$ there is no solution and the function $\Phi_i(x)$ is completely odd but translated, increasing and crossing the x axis once. Thus there is only one root x_i^*. To proceed to find the inflection points for $\frac{\partial^2 \Phi_i}{\partial x^2} = 0$ rewrite Eq. 47 to be:

$$\Phi_i(x) = \int \frac{dx}{\sqrt{1-\left[C_i(x-A_i)^2 + B_i\right]^2}} + c \quad \ldots\ldots\ldots\ldots\ldots\ldots (48)$$

Where:

$$A_i = \frac{\sum_{j=0}^{i}(P_j x_j - Q_j)}{\sum_{j=0}^{i} P_j}$$

$$B_i = -A_i^2 C_i + \frac{1}{EI_i}\sum_{j=0}^{i}(0.5P_j x_j^2 - Q_j x_j) + C1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (49)$$

$$C_i = \frac{1}{EI_i}\sum_{j=0}^{i} 0.5P_j$$

And from Eq. 25:

$$C1 = -\frac{1}{EI_{n-1}}\left[\sum_{j=0}^{n-1} 0.5P_j(L-x_j)^2 + Q_j(L-x_j)\right] \qquad (50)$$

Thus the inflection points are only three and they are:

$$\begin{aligned} x_{i1} &= A_i \\ x_{i2,3} &= A_i \pm \sqrt{-\frac{B_i}{C_i}} \end{aligned} \qquad (51)$$

The function becomes:

$$\Phi_i(x) = \int \frac{dx}{\sqrt{1-\left[C_i(x-A_i)^2 + B_i\right]^2}} + c = \int\left[1+\sum_{k=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\left[C_i(x-A_i)^2 + B_i\right]^{2k}\right]dx + c \qquad (52)$$

The slopes and the deflections become:

$$\dot{y}_i(x) = \frac{C_i(x-A_i)^2 + B_i}{\sqrt{1-\left[C_i(x-A_i)^2 + B_i\right]^2}} \quad \text{for} \quad x_i \le x \le x_{i+1} \qquad (53)$$

$$\begin{aligned} y_i(x) &= \int \frac{\left[C_i(x-A_i)^2 + B_i\right]dx}{\sqrt{1-\left[C_i(x-A_i)^2 + B_i\right]^2}} + C2_i \quad \text{for} \quad x_i \le x \le x_{i+1} \\ &= \frac{1}{3}C_i(x-A_i)^3 + B_i(x-A_i) + \int\left[\sum_{k=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\left[C_i(x-A_i)^2 + B_i\right]^{2k+1}\right]dx + C2_i \\ &= \int_{A_i}^{x} \frac{\left[C_i(z-A_i)^2 + B_i\right]dz}{\sqrt{1-\left[C_i(z-A_i)^2 + B_i\right]^2}} + C2_i \end{aligned} \qquad (54)$$

And find

$$\begin{aligned} C2_{n-1} &= -\int_{A_{n-1}}^{L} \dot{y}_n(x)\,dx \\ C2_{i-1} &= \int_{A_i}^{x_{i-1}} \dot{y}_i(x)\,dx - \int_{A_{i-1}}^{x_{i-1}} \dot{y}_{i-1}(x)\,dx + C2_i \end{aligned} \qquad (55)$$

And let

$$\psi_i(x_i) = \int_{x_i}^{x_{i+1}} \frac{dx}{\sqrt{1-\left[C_i(x-A_i)^2 + B_i\right]^2}} - L_i = \int_{x_i}^{x_{i+1}} \left[1 + \sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\left[C_i(x-A_i)^2 + B_i\right]^{2k}\right] dx - L_i$$

…………………………………………. (56)

For the numerical procedure, use Newton-Raphson method for systems of nonlinear algebraic equations. The following is the result for the Jacobian matrix. For a given function $\psi_i(x_i)$ from Eq. 56 the derivative with respect to x_m for $m = 1$ to n is:

$$\frac{\partial \psi_i(x_i)}{\partial x_m} = \left[\frac{\partial x}{\partial x_m} - A_{mi} - (x-A_i)\frac{B_{mi}}{2B_i}\right] \frac{1}{\sqrt{1-\left[C_i(x-A_i)^2 + B_i\right]^2}} \Bigg|_{x=x_i}^{x=x_{i+1}} + \frac{B_{mi}}{2B_i}(x_{i+1} - x_i)$$

$$+ \sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\left[2k + \frac{1}{2}\right] \frac{B_{mi}}{B_i} \int_{x_i}^{x_{i+1}} \left[C_i(x-A_i)^2 + B_i\right]^{2k} dx$$

$$= \left[\frac{\partial x}{\partial x_m} - A_{mi} - (x-A_i)\frac{B_{mi}}{2B_i}\right] \frac{1}{\sqrt{1-\left[C_i(x-A_i)^2 + B_i\right]^2}} \Bigg|_{x=x_i}^{x=x_{i+1}} + \frac{B_{mi}}{2B_i} \int_{x_i}^{x_{i+1}} \frac{1 + \left[C_i(x-A_i)^2 + B_i\right]^2}{\left[1 - \left[C_i(x-A_i)^2 + B_i\right]^2\right]^{\frac{3}{2}}} dx$$

……………………………………………………………. (57)

Where:

$$A_{mi} = \frac{P_m}{\sum_{j=0}^{i} P_j} \quad \text{for } m \le i \qquad \text{and}$$

$$A_{mi} = 0 \quad m > i$$

…………………….. (58)

$$B_{mi} = -2A_i A_{mi} C_i + \frac{P_m x_m - Q_m}{EI_i} + \frac{1}{EI_n}\left[P_m(L - x_m) + Q_m\right] \quad \text{for } m \le i \quad \text{and}$$

$$B_{mi} = \frac{1}{EI_n}\left[P_m(L - x_m) + Q_m\right] \qquad m > i$$

Thus, choose $x_i = \sum_{j=0}^{i} L_i$ for the initial condition and the Newton-Raphson method requires the updated $\{\bar{x}\} = \{x\} - [J]^{-1}\{\psi\}$. Where, $\{\bar{x}\}$ is the updated vector of $\bar{x}_i$, $\{x\}$ is the old vector of x_i, $[J]$ is the Jacobian matrix evaluated x_i and $\{\psi\}$ is the vector of function of Eq. 56 evaluated x_i.

Example:

In some cases this solution is the exact solution when the loads are actually point loads and moment for a beam. Setting up the solution for two point loads and two moments on a beam that is of two moment of inertia, see Fig. 8, and substituting yield,

$$C1 = -\frac{1}{EI_1}\left[0.5P_0(L-x_0)^2 + 0.5P_1(L-x_1)^2 + Q_0(L-x_0) + Q_1(L-x_1)\right]$$

$$A_0 = x_0 - \frac{Q_0}{P_0}$$

$$C_0 = \frac{0.5P_0}{EI_0}$$

$$B_0 = -A_0^2 C_0 + \frac{1}{EI_0}\left[0.5P_0 x_0^2 - Q_0 x_0\right] + C1$$

$$A_{00} = 1 \qquad A_{10} = 0$$

$$B_{00} = \frac{1}{EI_1}\left[P_0(L-x_0) + Q_0\right] \qquad B_{10} = \frac{1}{EI_1}\left(P_1(L-x_1) + Q_1\right)$$

$$A_1 = \frac{x_0 P_0 + x_1 P_1 - (Q_0 + Q_1)}{P_0 + P_1}$$

$$C_1 = \frac{0.5(P_0 + P_1)}{EI_0}$$

$$B_1 = -A_1^2 C_1 + \frac{1}{EI_1}\left[0.5P_0 x_0^2 + 0.5P_1 x_1^2 - (Q_0 x_0 + Q_1 x_1)\right] + C1$$

$$A_{01} = \frac{P_0}{P_0 + P_1} \qquad A_{11} = \frac{P_1}{P_0 + P_1}$$

$$B_{01} = \frac{P_0}{EI_1}\left[L - A_1\right] \qquad B_{10} = \frac{P_1}{EI_1}\left[L - A_1\right]$$

Let H consider the variables in Eq. 57

$$\frac{\partial \psi_0}{\partial x_m} = H_{m0}(x_0, x_1, A_0, B_0, C_0, A_{m0}, B_{m0})$$

$$\frac{\partial \psi_1}{\partial x_m} = H_{m1}(x_0, x_1, A_1, B_1, C_1, A_{m1}, B_{m1})$$

$$J = \begin{bmatrix} H_{00}(x_0, x_1, A_0, B_0, C_0, A_{00}, B_{00}) & H_{10}(x_0, x_1, A_0, B_0, C_0, A_{10}, B_{10}) \\ H_{11}(x_0, x_1, A_1, B_1, C_1, A_{01}, B_{01}) & H_{m1}(x_0, x_1, A_1, B_1, C_1, A_{11}, B_{11}) \end{bmatrix}$$

And an exact numerical solution is obtained.

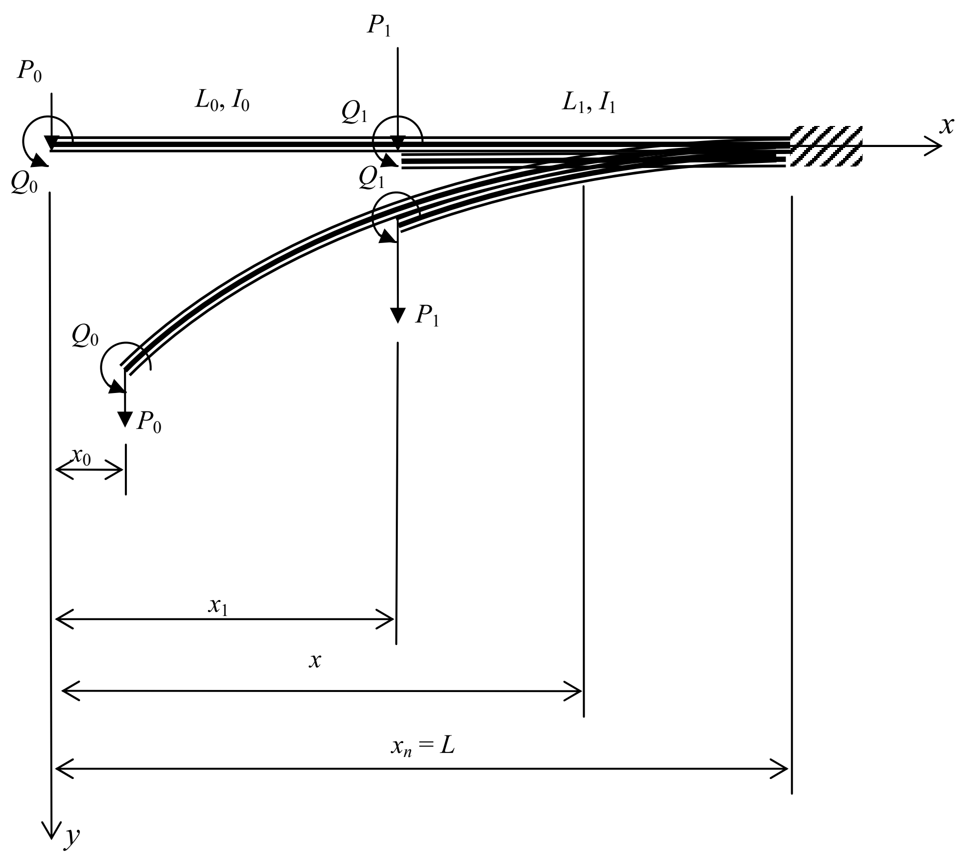

FIG.8 - Example

If for example the beam has to be divided to small increments due to the load function or the moment of inertia function, a less computational analysis may be selected as in Eq. 28.

APPENDIX B

Finding coefficients C1$_i$ with no discontinuity in the moment of inertia:

In Application for Case I, II, and III – (Numerical Solution for Any Load Function Non-Prismatic Beam including the examples), the moment of inertia at the joints were imposed equal. In the following equations this assumption will be shown valid and in Appendix C the derivation for discontinues beam for abrupt changes of the moment of inertia will be derived.

To start with the closed form solution of Eq.1 through Eq 18 will be used and the moment of inertia is taken as:

$$\left.\begin{array}{llll} \dfrac{1}{EI(x)} = \displaystyle\sum_{n=0}^{r} a_n x^n & \text{for Case I} & \text{defined in the interval} & 0 \le x \le L \\ \dfrac{1}{EI(y)} = \displaystyle\sum_{n=0}^{r} a_n y^n & \text{for Case II} & \text{defined in the interval} & 0 \le y \le y_0 \\ \dfrac{1}{EI(u)} = \displaystyle\sum_{n=0}^{r} a_n u^n & \text{for Case III} & \text{defined in the interval} & 0 \le u \le u_0 \end{array}\right] \Rightarrow n = 1, 2, \ldots\ldots, r \text{ and } r \ne \infty \qquad \ldots\ldots (59)$$

The following proof is for Case I. All other cases can be done with a similar proof.

From Eq. 3 the integral term for each segment becomes

$$Z_i(x) + C1_i = \int f(x)dx + C1_i = \int \left(\sum_{n=0}^{r} a_n x^n \right) \left(\sum_{j=0}^{i} P_j (x - x_j) + Q_j \right) dx + C1_i \quad \text{for} \quad x_i \le x \le x_{i+1} \qquad \ldots\ldots (60)$$

Using integration by parts starting with

$$u = \sum_{n=0}^{r} a_n x^n \quad \text{and} \quad dv = \sum_{j=0}^{i} P_j (x - x_j) + Q_j \quad \text{yeilds,}$$

$$Z_i(x) = \left(\sum_{n=0}^{r} a_n x^n\right)\left(\sum_{j=0}^{i} \frac{P_j (x - x_j)^2}{2} + Q_j (x - x_j)\right) + \int \left(\sum_{n=1}^{r} n a_n x^{n-1}\right)\left(\sum_{j=0}^{i} \frac{P_j (x - x_j)^2}{2} + Q_j (x - x_j)\right) dx$$

………………………………………………………. (61)

Continuing the integration by parts on each integral leads to

$$Z_i(x) = \sum_{k=0}^{r}\left\{\left(\sum_{n=k}^{r} \frac{n!}{(n-k)!} a_n x^{n-k}\right)\left(\sum_{j=0}^{i} \frac{P_j (x - x_j)^{k+2}}{(k+2)!} + \frac{Q_j (x - x_j)^{k+1}}{(k+1)!}\right)\right\} \quad \text{for} \quad x_i \le x \le x_{i+1}$$

……..…………………………… (62)

At x_i Eq. 3 for to consecutive segments at the joint becomes:

$$\dot{y}_i(x_i) = \frac{Z_i(x_i) + C1_i}{\sqrt{1 - [Z_i(x_i) + C1_i]^2}} \quad \text{and}$$

……………………………………………………………….. (63)

$$\dot{y}_{i-1}(x_i) = \frac{Z_{i-1}(x_i) + C1_{i-1}}{\sqrt{1 - [Z_{i-1}(x_i) + C1_{i-1}]^2}}$$

Apply compatibilities yields:

$$\underset{\text{Seg. Bm. } i-1 \quad \text{to} \quad \text{Seg. Bm. } i}{\dot{y}_{i-1}(x_i) = \dot{y}_i(x_i) \quad \text{at} \quad x = x_i}$$

…………………………………………………… (64)

Thus from Eq. 63

$$Z_{i-1}(x_i) + C1_{i-1} = Z_i(x_i) + C1_i$$ ……………………………………………………………… (65)

Substituting in Eq. 62 yields;

$$Z_{i-1}(x_i) = Z_i(x_i) = \sum_{k=0}^{r}\left\{\left(\sum_{n=k}^{r} \frac{n!}{(n-k)!} a_n x_i^{n-k}\right)\left(\sum_{j=0}^{i-1} \frac{P_j (x_i - x_j)^{k+2}}{(k+2)!} + \frac{Q_j (x_i - x_j)^{k+1}}{(k+1)!}\right)\right\}$$ ……..…….. (66)

Or,

$$C1_{i-1} = C1_i \quad \text{……} \tag{67}$$

Thus:

$$C1_0 = C1_1 = C1_2 = \cdots\cdots = C1_{n-1} = C1 \quad \text{……} \tag{68}$$

Thus, the coefficients C1$_i$ with no discontinuity in the moment of inertia is correct and the moment of inertia of the segment for Eq. 60 can be approximated as:

$$EI_{i+1} = \frac{\int_{l_i}^{l_{i+1}} I(x)}{L_{i+1}} \quad \text{where } l_i = \sum_{j=0}^{i} L_j \text{ and } l_{i+1} = \sum_{j=0}^{i+1} L_j \quad \text{……} \tag{69}$$

APPENDIX C

Finding coefficients C1$_i$ with discontinuity in the moment of inertia:

Application for Case I, II, and III are all similar and only Case I will be addressed. By using Eq. 64 and E. 65 in Eq. 22 the Z_i can be written at x_i as:

$$Z_{i-1} = \frac{1}{EI_{i-1}} \sum_{j=0}^{i-1} \left[0.5 P_j (x_i - x_j)^2 + Q_j (x_i - x_j) \right]$$

and ………………………………………… (70)

$$Z_i = \frac{1}{EI_i} \sum_{j=0}^{i-1} \left[0.5 P_j (x_i - x_j)^2 + Q_j (x_i - x_j) \right]$$

Thus from Eq. 65 yeilds;

$$C1_{i-1} = \left[\frac{1}{EI_i} - \frac{1}{EI_{i-1}} \right] \sum_{j=0}^{i-1} \left[0.5 P_j (x_i - x_j)^2 + Q_j (x_i - x_j) \right] + C1_i$$

or …………………………….. (71)

$$C1_i = \left[\frac{1}{EI_{i-1}} - \frac{1}{EI_i} \right] \sum_{j=0}^{i-1} \left[0.5 P_j (x_i - x_j)^2 + Q_j (x_i - x_j) \right] + C1_{i-1}$$

Starting with Eq. 25 yields;

$$C1_{n-1} = -\frac{1}{EI_{n-1}} \left[\sum_{j=0}^{n-1} 0.5 P_j (L - x_j)^2 + Q_j (L - x_j) \right] \quad \text{……………………………………… (72)}$$

Thus, all of the $C1_i$ can be found consecutively from Eq. 71 and Eq. 30 becomes:

$$L_i = \frac{x_{i+1} - x_i}{\sqrt{1 - \left[\frac{1}{EI_i} \left[\sum_{j=0}^{i} 0.5 P_j (x_i - x_j)^2 + Q_j (x_i - x_j) \right] + C1_i \right]^2}} \quad \text{………………………………… (73)}$$

So Eq. 31 becomes:

$$L_0 = \frac{x_1 - x_0}{\sqrt{1 - [C1_0]^2}} \quad \text{………………………………………………………………… (74).}$$

So the approximate numerical method with Eq. 28 starts by guessing $C1_0$ then find $(x_1 - x_0)$ from Eq. 74 and Eq. 71 with i = 1yields;

$$C1_1 = \left[\frac{1}{EI_0} - \frac{1}{EI_1}\right]\left[0.5P_0(x_1 - x_0)^2 + Q_0(x_1 - x_0)\right] + C1_0 \qquad (75)$$

So $C1_1$ can be found. By using Eq. 73 for i = 1yields;

$$L_1 = \frac{x_2 - x_1}{\sqrt{1 - \left[\frac{1}{EI_1}\left[0.5P_0(x_1 - x_0)^2 + Q_0(x_1 - x_0)\right] + C1_1\right]^2}} \qquad (76)$$

Now find $(x_2 - x_1)$ using Eq. 74, Eq. 75 and Eq. 76. , then find $(x_2 - x_0) = (x_2 - x_1) + (x_1 - x_0)$ and substitute in Eq. 71 yields;

$$C1_2 = \left[\frac{1}{EI_1} - \frac{1}{EI_0}\right]\left[0.5P_1(x_2 - x_1)^2 + Q_1(x_2 - x_1) + 0.5P_0(x_2 - x_0)^2 + Q_0(x_2 - x_0)\right] + C1_1 \qquad (77).$$

So all of the $C1_i$ then find $(x_{i+1} - x_i)$ can be found using Eq. 71 and Eq. 73. Now find $(x_n - x_0)$ from

$$(x_n - x_0) = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + ... + (x_1 - x_0) \qquad (78)$$

If $(x_n - x_0) = (L - x_0)$ then $x_0 = L - (x_n - x_0)$, and all of x_1 , x_2 , ……. , x_{n-1} can be found. By checking the end condition of Eq. 25 or Eq. 72 with Eq. 71

$$\dot{y}_n(L) = 0 \quad \Rightarrow \quad \frac{1}{EI_{n-1}}\left[\sum_{j=0}^{n-1} 0.5P_j(L-x_j)^2 + Q_j(L-x_j)\right] + C1_{n-1} = 0$$

or

$$C1_{n-1} = -\frac{1}{EI_{n-1}}\left[\sum_{j=0}^{n-1} 0.5P_j(L-x_j)^2 + Q_j(L-x_j)\right] \quad \text{since} \quad P_n = Q_n = 0$$

………………… (79)

check if

$$C1_{n-1} = \left[\frac{1}{EI_{n-2}} - \frac{1}{EI_{n-1}}\right]\sum_{j=0}^{n-1}\left[0.5P_j(x_{n-1}-x_j)^2 + Q_j(x_{n-1}-x_j)\right] + C1_{n-2}$$

If it is not satisfied update $C1_0$ with numerical analysis until all the variables are found. For the deflection from Eq. 4 yields:

$$y_i(x) = \int \frac{\frac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x-x_j)^2 + Q_j(x-x_j)\right] + C1_i}{\sqrt{1-\left[\frac{1}{EI_i}\left[\sum_{j=0}^{i} 0.5P_j(x-x_j)^2 + Q_j(x-x_j)\right] + C1_i\right]^2}}\,dx + C2_i \quad \text{for} \quad x_i \le x \le x_{i+1}$$

……………………….....…………………… (80)

To find $C2_i$ assume compatibility and enforce:

$y_n(x_n) = y_n(L) = 0$ and find $C2_{n-1}$
$y_{n-1}(x_{n-1}) = y_n(x_{n-1})$ and find $C2_{n-2}$
etc.....

And the solution is found numerically.

For using Newton-Raphson method in Appendix A, replace B_i in Eq. 43 by

$$A_i = \frac{\sum_{j=0}^{i}(P_j x_j - Q_j)}{\sum_{j=0}^{i} P_j}$$

$$B_i = -A_i^2 C_i + \frac{1}{EI_i}\sum_{j=0}^{i}(0.5P_j x_j^2 - Q_j x_j) + C1_i \quad \text{..(81)}$$

$$C_i = \frac{1}{EI_i}\sum_{j=0}^{i} 0.5P_j$$

Where $C1_i$ is from Eq. 71 and from Eq. 79

$$C1_{n-1} = -\frac{1}{EI_{n-1}}\left[\sum_{j=0}^{n-1} 0.5P_j (L - x_j)^2 + Q_j (L - x_j)\right] \quad \text{since} \quad P_n = Q_n = 0 \quad \text{.................... (82)}$$

So all of the $C1_i$ can be calculated from the vector {x} and update $\{\bar{x}\} = \{x\} - [J]^{-1}\{\psi\}$.

Where, $\{\bar{x}\}$ is the updated vector of $\bar{x}_i$, $\{x\}$ is the old vector of x_i, $[J]$ is the Jacobian matrix

evaluated x_i and $\{\psi\}$ is the vector of function of Eq. 50 evaluated x_i.

APPENDIX D

<u>Converting the Equations for Case I application to Elliptic functions</u>:

Eq. 26 and Eq. 27 in Case I applications can be re-written as in Eq. 48 yields:

$$L_i = \int_{x_i}^{x_{i+1}} \frac{dx}{\sqrt{1-\left[C_i(x-A_i)^2+B_i\right]^2}}$$

$$\dot{y}_i = \frac{C_i(x-A_i)^2+B_i}{\sqrt{1-\left[C_i(x-A_i)^2+B_i\right]^2}} \quad \text{.. (83)}$$

Where A_i , C_i and B_i are define in Eq. 49

Re-writing Eq. 77 to become:

$$L_i = \int_{x_i}^{x_{i+1}} \frac{dx}{\sqrt{1-\left[C_i(x-A_i)^2-(1-B_i)+1\right]^2}} \quad \text{... (84)}$$

Now let

$$\sqrt{C_i}(x-A_i) = \sqrt{1-B_i}\cos\phi \qquad \text{with} \qquad \sqrt{C_i}dx = -\sqrt{1-B_i}\sin\phi$$

and .... (85)

$$C_i(x-A_i)^2 = (1-B_i)\cos^2\phi$$

$$L_i = -\sqrt{\frac{1-B_i}{C_i}}\int_{\varphi 1_i}^{\phi 2_{i+1}} \frac{\sin\phi}{\sqrt{1-\left[-(1-B_i)\sin^2\phi+1\right]^2}}d\phi \quad \text{... (86)}$$

Now multiply the square in the denominator in Eq. 86 and rearrange yield:

$$L_i = -\frac{1}{\sqrt{2C_i}}\int_{\phi 1_i}^{\phi 2_{i+1}} \frac{d\phi}{\sqrt{1-\left(\sqrt{\frac{1-B_i}{2}}\right)^2\sin^2\phi}} \quad \text{... (87)}$$

Where:

$$\phi 1_i = \cos^{-1}\left[\sqrt{\frac{C_i}{1-B_i}}(x_i - A_i)\right] = \cos^{-1}\left[\frac{\sqrt{C_i/2}}{p_i}(x_i - A_i)\right]$$

and .... (88)

$$\phi 2_{i+1} = \cos^{-1}\left[\sqrt{\frac{C_i}{1-B_i}}(x_{i+1} - A_i)\right] = \cos^{-1}\left[\frac{\sqrt{C_i/2}}{p_i}(x_{i+1} - A_i)\right]$$

Where

$$p_i = \sqrt{\frac{1-B_i}{2}} \qquad \text{(89)}$$

And Eq. 87 becomes:

$$L_i = -\frac{1}{\sqrt{2C_i}}\int_{\phi 1_i}^{\phi 2_{i+1}} \frac{d\phi}{\sqrt{1-p_i^2 \sin^2\phi}} \qquad \text{(90)}$$

Which can be expressed in Elliptic Integral as follows:

$$L_i = \frac{1}{\sqrt{2C_i}}\left[F(p_i, \phi 2_{i+1}) - F(p_i, \phi 1_i)\right] \qquad \text{(91)}$$

Where the function F is the elliptic integral of the first kind.

Similarly Eq. 83 becomes:

$$\dot{y}_i(x) = \frac{C_i(x-A_i)^2 - (1-B_i) + 1}{\sqrt{1-\left[C_i(x-A_i)^2 - (1-B_i) + 1\right]^2}} \qquad \text{(92)}$$

Now substitute Eq. 89 in 92 yields:

$$\dot{y}_i(x) = \frac{-2p_i^2 \sin^2\phi + 1}{2p_i \sin\phi\sqrt{1-p_i^2\sin^2\phi}}$$

where .... (93)

$$x = \frac{p_i}{\sqrt{C_i/2}}\cos\phi + A_i$$

The deflection between the joints becomes:

$$\int_{y_i}^{y_{i+1}} \dot{y}dx = y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} \frac{C_i(x-A_i)^2 - (1-B_i)+1}{\sqrt{1-\left[C_i(x-A_i)^2-(1-B_i)+1\right]^2}}\,dx \quad \text{.................................... (94)}$$

Eq. 94 can be re-written and using Eq.84 yields:

$$\begin{aligned} y_{i+1} - y_i &= \int_{x_i}^{x_{i+1}} \frac{C_i(x-A_i)^2 - (1-B_i)}{\sqrt{1-\left[C_i(x-A_i)^2-(1-B_i)+1\right]^2}}\,dx + \int_{x_i}^{x_{i+1}} \frac{1}{\sqrt{1-\left[C_i(x-A_i)^2-(1-B_i)+1\right]^2}}\,dx \\ &= \int_{x_i}^{x_{i+1}} \frac{C_i(x-A_i)^2 - (1-B_i)}{\sqrt{1-\left[C_i(x-A_i)^2-(1-B_i)+1\right]^2}}\,dx + L_i \end{aligned}$$

$$\text{.. (95)}$$

Now substitute Eq. 89 in 95 yields:

$$\begin{aligned} y_{i+1} - y_i &= L_i + \frac{\sqrt{1-B_i}}{\sqrt{C_i}} \int_{\phi 1_i}^{\phi 2_{i+1}} \frac{p_i^2 \sin^2\phi}{p_i\sqrt{1-p_i^2\sin^2\phi}}\,d\phi \\ &= L_i + \frac{1}{\sqrt{C_i/2}} \int_{\phi 1_i}^{\phi 2_{i+1}} \frac{p_i^2 \sin^2\phi}{\sqrt{1-p_i^2\sin^2\phi}}\,d\phi \\ &= L_i + \frac{2}{\sqrt{2C_i}}\left[\int_{\phi 1_i}^{\phi 2_{i+1}} \frac{1}{\sqrt{1-p_i^2\sin^2\phi}}\,d\phi - \int_{\phi 1_i}^{\phi 2_{i+1}} \sqrt{1-p_i^2\sin^2\phi}\,d\phi\right] \\ &= -L_i - \frac{2}{\sqrt{2C_i}} \int_{\phi 1_i}^{\phi 2_{i+1}} \sqrt{1-p_i^2\sin^2\phi}\,d\phi = -L_i - \sqrt{\frac{2}{C_i}} E(p_i, \phi 2_{i+1}) - E(p_i, \phi 1_i) \end{aligned} \quad \text{......... (96)}$$

Where the function E is the elliptic integral of the second kind.

APPENDIX E

Other End Conditions:

<u>End Condition #1</u>: This condition has a pin at bottom end and rotational fixed at the top end but free to translate at the top as in Fig. 9.

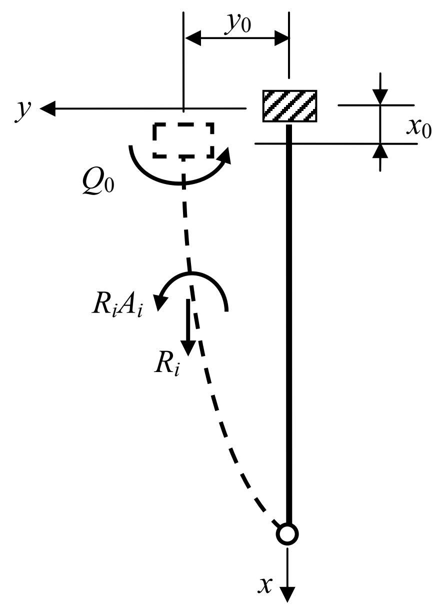

Fig. 9 Pin – Rotational Fixed Column

Thus the moment Q_0 at the tip of the column makes $\left.\frac{dy}{dx} = 0\right|_{@\, y=x=0}$ Now at $y = y_n = 0$ and $x = L$

$f_n(L) = 0$ since the moment at the bottom is zero or Eq. 21 becomes:

$$f_n(L) = \frac{M_{n-1}}{EI_{n-1}} = \frac{1}{EI_{n-1}}\left[\sum_{j=0}^{n-1} P_j(L - x_j) + Q_j\right] = 0$$

or

$$Q_0 + P_0(L - x_0) + \sum_{j=1}^{n-1} P_j(L - x_j) + Q_j = 0 \quad \text{……………………..……….. (97)}$$

or

$$Q_0 = -P_0(L - x_0) - \sum_{j=1}^{n-1}\left(P_j(L - x_j) + Q_j\right)$$

Let

$$R_i = \sum_{j=0}^{i} P_j \quad \text{…………………………………………………………..……………….. (98)}$$

Then from Eq. 49

$$A_0 = \frac{Q_0}{R_0} = \frac{1}{R_0}\left\{-P(L - x_0) - \sum_{j=1}^{n-1}\left[P(L - x_j) + Q_j\right]\right\} \quad \text{……………………………….…….. (99)}$$

Thus $\frac{dy}{dx} = 0\Big|_{@\, y=x=0} \rightarrow$ from Eq. 83 $C_0(A_0)^2 = -B_0$

And Eq. 24 does not apply. So rewriting Eq. 23 using Eq. 83 to

$$C_{i-1}(x_i - A_{i-1})^2 + B_{i-1} = C_i(x_i - A_i)^2 + B_i$$

or ……………………………………………… (100)

$$B_i = -C_i(x_i - A_i)^2 + C_{i-1}(x_i - A_{i-1})^2 + B_{i-1}$$

Therefore by substituting A_i , A_{i-1} , B_{i-1} in Eq. 99 then B_i is found and $\phi 2_{i+1}$, $\phi 1_i$ and p_i are found from Eq. 88 and Eq. 89 and the problem is solved using the elliptical integral.

Note: Eq. 56 and 57 can be written as:

$$\psi_i(x_i) = -\frac{1}{\sqrt{2C_i}}\int_{\phi 1_i}^{\phi 2_{i+1}} \frac{d\phi}{\sqrt{1 - p_i^2 \sin^2\phi}} - L_i \quad \text{…………………………………………….. (101)}$$

$$\frac{\partial \psi_i(x_i)}{\partial x_m} = \left[\frac{\partial x}{\partial x_m} - A_{mi} - (x - A_i)\frac{B_{mi}}{2B_i}\right]\frac{1}{\sqrt{1-\left[C_i(x-A_i)^2 + B_i\right]^2}}\Bigg|_{x=x_i}^{x=x_{i+1}} + \frac{B_{mi}}{2B_i}(x_{i+1} - x_i)$$

$$+\sum_{k=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\left[2k+\frac{1}{2}\right]\frac{B_{mi}}{B_i}\int_{x_i}^{x_{i+1}}\left[C_i(x-A_i)^2 + B_i\right]^{2k} dx$$

$$= \left[\frac{\partial x}{\partial x_m} - A_{mi} - (x - A_i)\frac{B_{mi}}{2B_i}\right]\frac{1}{\sqrt{1-\left[C_i(x-A_i)^2 + B_i\right]^2}}\Bigg|_{x=x_i}^{x=x_{i+1}} + \frac{B_{mi}}{4B_i p_i^2\sqrt{2C_i}}\int_{\phi 1_i}^{\phi 2_{i+1}}\frac{1+\left(\cos^2\phi + B_i\sin^2\right)^2}{\sin^2\phi\left[1-p_i^2\sin^2\phi\right]^{\frac{3}{2}}}\,d\phi$$

……………………………………………………………… (102)

End Condition #2 Same as condition #1 except fixed at $x = L$ as in Fig 10.

Using Eq. 93 at $i = n$, $x_n = L \rightarrow \frac{dy}{dx} = 0\Big|_{@\,x=L}$ yields:

$$C_{i-1}(L - A_{i-1})^2 + B_{i-1} = 0 \qquad \text{……………………………………………………………… (103)}$$

For $f_n(L) = 0$ since the total moment at the bottom is zero implies

$$f_n(L) = \frac{M_{n-1}}{EI_{n-1}} = \frac{1}{EI_{n-1}}\left[\sum_{j=0}^{n-1} P_j(L - x_j) + Q_j - Q_n\right] = 0$$

or

$$Q_0 + P_0(L - x_0) + \sum_{j=1}^{n-1} P_j(L - x_j) + Q_j - Q_n = 0 \qquad \text{………………………………………….(104)}$$

or

$$Q_0 = -P_0(L - x_0) - \sum_{j=1}^{n-1}\left(P_j(L - x_j) + Q_j\right) + Q_n$$

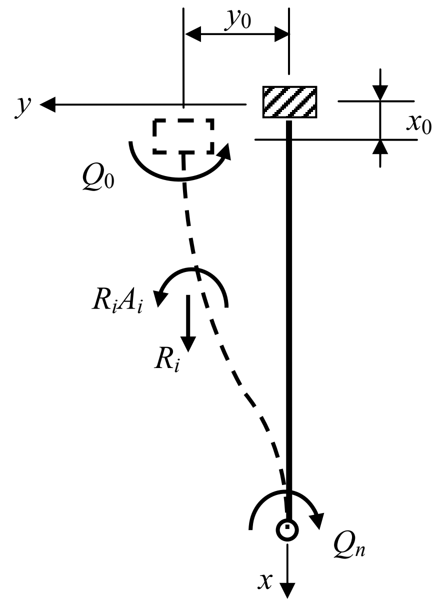

Fig. 10 Fixed – Rotational Fixed Column

Thus the procedure is to pick Q_n and find Q_0 from Eq. 97 and use procedure in condition #1 above to solve for y_i for given L_i and find all B_i then $\phi 1_{i+1}$, $\phi 2_i$ and p_i are from Eq. 82 and Eq. 83 and check if Eq. 96 is satisfied if not update Q_n and the problem is solved using the elliptical integral.

End Condition #3 Pined both ends as in Fig 11.

Subdividing the column into two parts Part A and Part B at point E at line $a - a$ where the slope $\frac{dy}{dx} = 0$ in which it is to be found. Separate the loads of Part A and Part B and solve for y_{0A} for Part A using a straight cantilever column fixed at point E. Then solve Part B using end condition #1 (pin at bottom end and rotational fixed at the top end but free to translate at the top.) and find y_{0B} . Check and see if $y_{0A} = y_{0B}$. If y_{0A} not equal to y_{0B} then move point E at line $a - a$ up if $y_{0A} > y_{0B}$ or down if $y_{0A} < y_{0B}$ until point E is found.

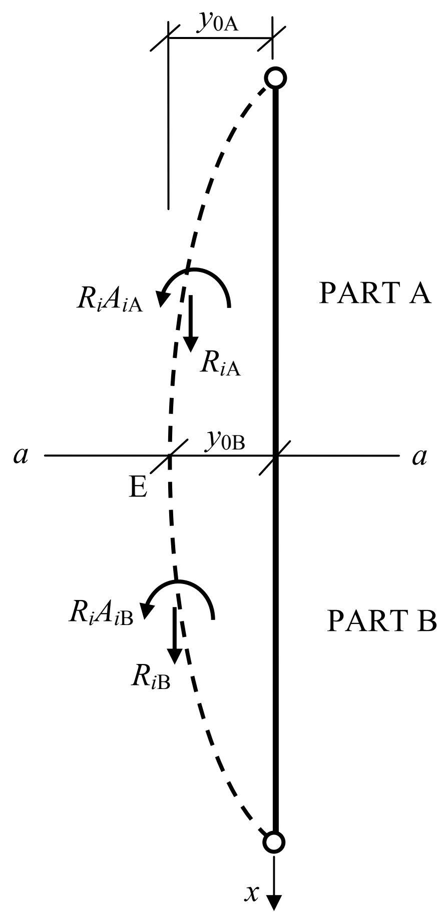

Fig 11 – Pined both ends.

End Condition #4 Fixed both ends.

This condition is similar to end condition #3 where Part A and B are as condition #2.