Exact and Numerical Solution for Large Deflection of Elastic Non-Prismatic Plates

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General Solution for symmetry Case I:
Staring with Eq. 37 and 38 pp 39 and Eq 102 pp 81 from "Theory of Plates and Shells" by Timoshenko and Woinowsky-Krieger we have:

$$M_x = D\left[\frac{1}{r_x} + \nu\frac{1}{r_y}\right]$$

$$M_y = D\left[\frac{1}{r_y} + \nu\frac{1}{r_x}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

Thus:

$$M_x + M_y = (1+\nu)D\left[\frac{1}{r_x} + \frac{1}{r_y}\right]$$

$$or \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

$$\frac{1}{r_x} + \frac{1}{r_y} = \frac{M_x + M_y}{(1+\nu)D} = f(x+y)$$

Where $f(x + y)$ is a function to be found, in here we assume symmetry and the moments are a function on $x + y$ only.

As a result we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial (x+y)}\frac{\partial (x+y)}{\partial x} = \frac{\partial w}{\partial (x+y)}$$

$$and \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial (x+y)^2}$$

Let $m = x + y$ and substitute in Eq. 2 we have:

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$$\frac{1}{r_x} = \frac{1}{r_y} = \frac{1}{r_m}$$

and ………………………………………………….. (4)

$$\frac{1}{r_n} = \frac{-\dfrac{\partial^2 w}{\partial m^2}}{\left[1+\left(\dfrac{\partial w}{\partial m}\right)^2\right]^{\frac{3}{2}}} = \frac{f(m)}{2}$$

The solution of Eq. 4 from before:

$$w = \mp\int \frac{\int 0.5 f(m)dm + C_1}{\sqrt{1-\left(\int 0.5 f(m)dm + C_1\right)^2}}\, dm + C_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

And

$$w_x = w_y = \mp \frac{\int 0.5 f(m)dm + C_1}{\sqrt{1-\left(\int 0.5 f(m)dm + C_1\right)^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

And

$$M_x = D\left[\frac{1}{r_x} + \nu\frac{1}{r_y}\right] = 0.5D(1+\nu)f(m)$$

$$M_y = D\left[\frac{1}{r_y} + \nu\frac{1}{r_x}\right] = 0.5D(1+\nu)f(m) \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = D(1-\nu)\frac{\partial^2 w}{\partial m^2} = -\frac{0.5D(1-\nu)f(m)}{\left[1-\left(\int 0.5 f(m)dm + C_1\right)^2\right]^{\frac{3}{2}}}$$

From Timosheko Eq81 pp100 we have:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -q(x+y) = -q(m)$$

or ……………………………….. (8)

$$M_x + M_y - 2M_{xy} = -\iint q(m)dmdm = -M_{T+}(m)$$

Where M_{T+} is the total moment on the plate segment

Substituting Eq. 7 in Eq. 8 and rearranging we have:

$$D(1+\nu)f(m) + \frac{D(1-\nu)f(m)}{\left[1-\left(\int 0.5f(m)dm + C_1\right)^2\right]^{\frac{3}{2}}} = -M_{T+}(m)$$

or …………………...…… (9)

$$0.5f(m) + \frac{1-\nu}{1+\nu}\frac{0.5f(m)}{\left[1-\left(\int 0.5f(m)dm + C_1\right)^2\right]^{\frac{3}{2}}} = -\frac{M_{T+}(m)}{2(1+\nu)D}$$

Let:

$$u = \int 0.5f(m)dm + C_1 \quad \text{…………………………………………..……….} \; (10)$$

And integrate Eq. 9 with respect to n yields:

$$u + \left(\frac{1-\nu}{1+\nu}\right)\frac{u}{\sqrt{1-u^2}} = -\int \frac{M_{T+}(m)}{2(1+\nu)D}dm \quad \text{………………………………………. } (11)$$

Eq. 11 is a Quartic equation in u and has four roots, two are for coordinates (x, y) and (y, x) and the others are possibly imaginary.

And the solution is found numerically by substituting back in Eq. 5, 6 and 7. An alternative preferred solution for Eq. 11 can be found by expressing in $\dot{w} = w_x = w_y$. Rewriting Eq. 6 to

$$\dot{w} = \mp \frac{u}{\sqrt{1-u^2}}$$

or ………………………………………………………………… (12)

$$u = -\frac{\dot{w}}{\sqrt{1+\dot{w}^2}}$$

Substitute Eq.12 in Eq. 11 yields:

$$\frac{\dot{w}}{\sqrt{1+\dot{w}^2}}+\left(\frac{1-\nu}{1+\nu}\right)\dot{w}=\int\frac{M_{T+}(m)}{2(1+\nu)D}dm \quad \ldots\ldots (13)$$

Let $\dot{w}=\tan\theta$, so it is defined everywhere, and substitute in Eq. 13 yeilds

$$\sin\theta+\left(\frac{1-\nu}{1+\nu}\right)\tan\theta=\int\frac{M_{T+}(m)}{2(1+\nu)D}dm \quad \ldots\ldots (14)$$

Equation 14 has two basic roots in θ and the rest are differs by 2π multiples.

Thus $\dot{w}$ of Eq. 13 has only two real roots for coordinates (x, y) and (y, x).

<u>General Solution for axi-symmetry Case II</u>:
Changing Eq. 2 to a function on $x-y$, thus

$$M_x+M_y=(1+\nu)D\left[\frac{1}{r_x}+\frac{1}{r_y}\right]$$
$$or \quad \ldots\ldots (15)$$
$$\frac{1}{r_x}+\frac{1}{r_y}=\frac{M_x+M_y}{(1+\nu)D}=g(x-y)$$

Where $g(x - y)$ is a function to be found, in here we assume axi-symmetry and the moments are a function on $x - y$ only.

As a result we have

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial(x-y)}\frac{\partial(x-y)}{\partial x}=\frac{\partial w}{\partial(x-y)}$$
$$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial(x-y)}\frac{\partial(x-y)}{\partial y}=-\frac{\partial w}{\partial(x-y)} \quad \ldots\ldots (16)$$
$$and$$
$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w}{\partial y^2}=\frac{\partial^2 w}{\partial x\partial y}=\frac{\partial^2 w}{\partial(x-y)^2}$$

Let $n = x - y$ and substitute in Eq. 15 we have:

$$\frac{1}{r_x} = \frac{1}{r_y} = \frac{1}{r_n}$$

and ……………………………………………….. (17)

$$\frac{1}{r_n} = \frac{-\frac{\partial^2 w}{\partial n^2}}{\left[1+\left(\frac{\partial w}{\partial n}\right)^2\right]^{\frac{3}{2}}} = \frac{g(n)}{2}$$

The solution of Eq. 17 from before:

$$w = \mp\int \frac{\int 0.5g(n)dn + C_1}{\sqrt{1-\left(\int 0.5f(n)dn + C_1\right)^2}}\, dn + C_2 \quad \text{……………………………………} (18)$$

And

$$w_x = -w_y = \mp\frac{\int 0.5g(n)dn + C_1}{\sqrt{1-\left(\int 0.5g(n)dn + C_1\right)^2}} \quad \text{………………………………………} (19)$$

And

$$M_x = D\left[\frac{1}{r_x} + \nu\frac{1}{r_y}\right] = 0.5D(1+\nu)g(n)$$

$$M_y = D\left[\frac{1}{r_y} + \nu\frac{1}{r_x}\right] = 0.5D(1+\nu)g(n) \quad \text{…………..} (20)$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = -D(1-\nu)\frac{\partial^2 w}{\partial n^2} = \frac{0.5D(1-\nu)g(n)}{\left[1-\left(\int 0.5g(n)dn + C_1\right)^2\right]^{\frac{3}{2}}}$$

From Timosheko Eq81 pp100 we have:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -q(x-y) = -q(n)$$

or

$$\frac{\partial^2 M_x}{\partial n^2} + \frac{\partial^2 M_y}{\partial n^2} + 2\frac{\partial^2 M_{xy}}{\partial n^2} = -q(x-y) = -q(n) \quad \text{.................................. (21)}$$

or

$$M_x + M_y + 2M_{xy} = -\iint q(n)dndn = -M_{T-}(n)$$

Where M_{T-} is the total moment on the plate segment

Substituting Eq. 19 in Eq. 21 and rearranging we have:

$$D(1+\nu)g(n) + \frac{D(1-\nu)g(n)}{\left[1-\left(\int 0.5g(n)dn + C_1\right)^2\right]^{\frac{3}{2}}} = -M_{T-}(n)$$

or .... (22)

$$0.5g(n) + \frac{1-\nu}{1+\nu}\frac{0.5g(n)}{\left[1-\left(\int 0.5g(n)dn + C_1\right)^2\right]^{\frac{3}{2}}} = -\frac{M_{T-}(n)}{2(1+\nu)D}$$

Let:

$$v = \int 0.5g(n)dn + C_1 \quad \text{.. (23)}$$

And integrate Eq. 22 with respect to *n* yields:

$$v + \left(\frac{1-\nu}{1+\nu}\right)\frac{v}{\sqrt{1-v^2}} = -\int \frac{M_{T-}(n)}{2(1+\nu)D}dn \quad \text{... (24)}$$

Eq. 24 is a Quartic equation in *v* and has four roots, two are for coordinates (*x*, -*y*) and (*y*, -*x*) and the others are possibly imaginary.
And the solution is found numerically by substituting back in Eq. 18, 19 and 20.

An alternative preferred solution for Eq. 24 can be found by expressing in $\dot{w} = w_x = -w_y$. Rewriting Eq. 19 to

$$\dot{w} = \mp \frac{v}{\sqrt{1-v^2}}$$

or ……………………………………………………………… (25)

$$v = -\frac{\dot{w}}{\sqrt{1+\dot{w}^2}}$$

Substitute Eq.25 in Eq. 24 yields:

$$\frac{\dot{w}}{\sqrt{1+\dot{w}^2}} + \left(\frac{1-\nu}{1+\nu}\right)\dot{w} = \int \frac{M_{T-}(n)}{2(1+\nu)D} dn \quad \text{(26)}$$

Let $\dot{w} = \tan\theta$, so it is defined everywhere, and substitute in Eq. 13 yeilds

$$\sin\theta + \left(\frac{1-\nu}{1+\nu}\right)\tan\theta = \int \frac{M_{T-}(n)}{2(1+\nu)D} dn \quad \text{(27)}$$

Equation 27 has two basic roots in θ and the rest are differs by 2π multiples.

Thus $\dot{w}$ of Eq. 26 has only two real roots for coordinates (x,- y) and (y,- x).

General Solution for Case III:
Changing Eq. 2 to a function on $ax + by$, thus

$$M_x + M_y = (1+\nu)D\left[\frac{1}{r_x} + \frac{1}{r_y}\right]$$

or ………………………………………………. (28)

$$\frac{1}{r_x} + \frac{1}{r_y} = \frac{M_x + M_y}{(1+\nu)D} = h(ax+by)$$

Where $h(ax + by)$ is a function to be found, in here we assume the moments are a function on $ax + by$ only.

As a result we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial(ax+by)}\frac{\partial(ax+by)}{\partial x} = a\frac{\partial w}{\partial(ax+by)}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial(ax+by)}\frac{\partial(ax+by)}{\partial y} = b\frac{\partial w}{\partial(ax+by)}$$

and

$$\frac{\partial^2 w}{\partial x^2} = a^2\frac{\partial^2 w}{\partial(ax+by)^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(29)}$$

$$\frac{\partial^2 w}{\partial y^2} = b^2\frac{\partial^2 w}{\partial(ax+by)^2}$$

$$\frac{\partial^2 w}{\partial x\partial y} = ab\frac{\partial^2 w}{\partial(ax+by)^2}$$

Let $t = ax + by$ and substitute in Eq. 2 we have:

$$\frac{1}{r_x}+\frac{1}{r_y} = \frac{-a^2\dfrac{\partial^2 w}{\partial t^2}}{\left[1+a^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}} + \frac{-b^2\dfrac{\partial^2 w}{\partial t^2}}{\left[1+b^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}} = h(t) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(30)}$$

And

$$M_x = D\left[\frac{1}{r_x}+\nu\frac{1}{r_y}\right] = D\left[\frac{-a^2\dfrac{\partial^2 w}{\partial t^2}}{\left[1+a^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}} + \frac{-\nu b^2\dfrac{\partial^2 w}{\partial t^2}}{\left[1+b^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}}\right]$$

$$M_y = D\left[\frac{1}{r_y}+\nu\frac{1}{r_x}\right] = D\left[\frac{-b^2\dfrac{\partial^2 w}{\partial t^2}}{\left[1+b^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}} + \frac{-\nu a^2\dfrac{\partial^2 w}{\partial t^2}}{\left[1+a^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}}\right] \qquad \ldots\ldots\ldots\ldots \text{(31)}$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x\partial y} = D(1-\nu)ab\frac{\partial^2 w}{\partial t^2}$$

From Timosheko Eq81 pp100 we have:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -q(ax+by) = -q(t)$$

or

$$a^2\frac{\partial^2 M_x}{\partial t^2} + b^2\frac{\partial^2 M_y}{\partial t^2} - 2ab\frac{\partial^2 M_{xy}}{\partial t^2} = -q(ax+by) = -q(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (32)$$

or

$$a^2 M_x + b^2 M_y - 2abM_{xy} = -\iint q(t)\,dt\,dt = -M_T(t)$$

Where M_T is the total moment on the plate segment

Substituting Eq. 31 in Eq. 32 and rearranging we have:

$$Da^2\left[\frac{-a^2\dfrac{\partial^2 w}{\partial t^2}}{\left[1+a^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}} + \frac{-\nu b^2\dfrac{\partial^2 w}{\partial t^2}}{\left[1+b^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}}\right] + Db^2\left[\frac{-b^2\dfrac{\partial^2 w}{\partial t^2}}{\left[1+b^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}} + \frac{-\nu a^2\dfrac{\partial^2 w}{\partial t^2}}{\left[1+a^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}}\right]$$

$$-2D(1-\nu)ab\frac{\partial^2 w}{\partial t^2} = -M_T(t)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (33)$$

Or

$$\frac{(a^4+\nu a^2 b^2)\dfrac{\partial^2 w}{\partial t^2}}{\left[1+a^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}} + \frac{(b^4+\nu a^2 b^2)\dfrac{\partial^2 w}{\partial t^2}}{\left[1+b^2\left(\dfrac{\partial w}{\partial t}\right)^2\right]^{\frac{3}{2}}} + 2(1-\nu)ab\frac{\partial^2 w}{\partial t^2} = \frac{1}{D}M_T(t)$$

or $\ldots\ldots\ldots\ldots\ldots$ (34)

$$\frac{(a^4+\nu a^2 b^2)\ddot{w}}{\left[1+a^2\dot{w}^2\right]^{\frac{3}{2}}} + \frac{(b^4+\nu a^2 b^2)\ddot{w}}{\left[1+b^2\dot{w}^2\right]^{\frac{3}{2}}} + 2(1-\nu)ab\ddot{w} = \frac{1}{D}M_T(t)$$

Let

$$\dot{w} = \tan\theta \quad \text{so} \quad \ddot{w} = \frac{\dot{\theta}}{\cos^2\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(35)}$$

Substitute Eq. 35 in Eq. 34 yields;

$$\frac{(a^4 + va^2b^2)\cos\theta\dot{\theta}}{\left[\cos^2\theta + a^2\sin^2\theta\right]^{\frac{3}{2}}} + \frac{(b^4 + va^2b^2)\cos\theta\dot{\theta}}{\left[\cos^2\theta + b^2\sin^2\theta\right]^{\frac{3}{2}}} + 2(1-\nu)ab\frac{\dot{\theta}}{\cos^2\theta} = \frac{1}{D}M_T(t)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(36)}$$

$$\frac{(a^4 + va^2b^2)\sin\theta}{\sqrt{1+(a^2-1)\sin^2\theta}} + \frac{(b^4 + va^2b^2)\sin\theta}{\sqrt{1+(b^2-1)\sin^2\theta}} + 2(1-\nu)ab\tan\theta = \frac{1}{D}\int M_T(t)dt \ldots\ldots\ldots\ldots \text{(37)}$$

Equation 37 has two basic roots in θ and the rest are differs by 2π multiples. Thus $\dot{w}$ of Eq. 35 has only two real roots for coordinates (*x/a*, *y/b*) and (*y/a*, *x/b*).

<u>General Solution Case IV</u>:
The following is the general solution for vertical loads on the plate without buckling in the Cartesian coordinates using Taylor's and Fourier's representation.

We start by rewriting the partial differential (Eq. 100 pp81 Timoshenko and Woinowsky-Krieger book using Eq. 37, 35 and Eq. 102 pp81) as follows:

$$\frac{\partial^2}{\partial x^2}\left[\frac{1}{r_x} + \nu\frac{1}{r_y}\right] + \frac{\partial^2}{\partial y^2}\left[\frac{1}{r_y} + \nu\frac{1}{r_x}\right] - 2(1-\nu)\frac{\partial^4 w}{\partial x^2 \partial y^2} = -\frac{1}{D}q(x,y) \ \ldots\ldots\ldots\ldots\ldots\ldots \text{(38)}$$

Eq. 38 can be written as:

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial x}\left(\frac{w_x}{\sqrt{1+w_x^2}}\right) + \nu\frac{\partial}{\partial y}\left(\frac{w_y}{\sqrt{1+w_y^2}}\right)\right] \\
&\qquad + \frac{\partial^2}{\partial y^2}\left[\frac{\partial}{\partial y}\left(\frac{w_y}{\sqrt{1+w_y^2}}\right) + \nu\frac{\partial}{\partial x}\left(\frac{w_x}{\sqrt{1+w_x^2}}\right)\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(39)} \\
&\qquad\qquad + (1-\nu)\frac{\partial^3 w_x}{\partial y^2 \partial x} + (1-\nu)\frac{\partial^3 w_y}{\partial x^2 \partial y} = \frac{1}{D}q(x,y)
\end{aligned}$$

Let us express w with an accurate approximation using Taylor expansion polynomial over the intervals $0 \le x \le a$ and $0 \le y \le b$ for a rectangular portion of the plate (this representation has to exist since we are not assuming plates deflecting to infinity and from physics for every load there is a unique deflection to be guaranteed in the elastic realm giving a certain bounded function over that portion of the plate. Thus, the function can be represented by a Taylor expansion polynomial using Taylor theorem with unique coefficients for every load) so:

$$w = \sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk} x^j y^k \quad \ldots\ldots (40)$$

Thus

$$w_x = \sum_{j=1}^{r}\sum_{k=0}^{t} j p_{jk} x^{j-1} y^k \quad \ldots\ldots (41)$$

$$w_y = \sum_{j=0}^{r}\sum_{k=1}^{t} k p_{jk} x^j y^{k-1} \quad \ldots\ldots (42)$$

Let:

$$w_x = \frac{g(x,y)}{\sqrt{1-[g(x,y)]^2}} \quad \ldots\ldots (43)$$

$$w_y = \frac{h(x,y)}{\sqrt{1-[h(x,y)]^2}} \quad \ldots\ldots (44)$$

Then

$$g(x,y) = \frac{w_x}{\sqrt{1+w_x^2}} = \frac{\sum_{j=1}^{r}\sum_{k=0}^{t} j p_{jk} x^{j-1} y^k}{\sqrt{1+\left[\sum_{j=1}^{r}\sum_{k=0}^{t} j p_{jk} x^{j-1} y^k\right]^2}} \quad \ldots\ldots (45)$$

$$h(x,y) = \frac{w_y}{\sqrt{1+w_y^2}} = \frac{\sum_{j=0}^{r}\sum_{k=1}^{t} k p_{jk} x^j y^{k-1}}{\sqrt{1+\left[\sum_{j=0}^{r}\sum_{k=1}^{t} k p_{jk} x^j y^{k-1}\right]^2}} \quad \ldots\ldots (46)$$

Now express w_x, w_y, $g(x,y)$, $h(x,y)$, $q(x,y)$ and in Fourier's representation over the interval $0 \le x \le a$ and $0 \le y \le b$ for a rectangular portion of the plate in Cartesian coordinates as follows:

$$w_x = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} w_{xmn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad \ldots\ldots (47)$$

$$g(x,y)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} g_{mn}\cos\frac{m\pi\, x}{a}\sin\frac{n\pi\, y}{b} \quad \text{...} \quad (48)$$

$$w_y=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} w_{ymn}\sin\frac{m\pi\, x}{a}\cos\frac{n\pi\, y}{b} \quad \text{...} \quad (49)$$

$$h(x,y)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} h_{mn}\sin\frac{m\pi\, x}{a}\cos\frac{n\pi\, y}{b} \quad \text{...} \quad (50)$$

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin\frac{m\pi\, x}{a}\sin\frac{n\pi\, y}{b} \quad \text{...} \quad (51)$$

Where:

$$w_{xmn}=\frac{4}{ab}\int_0^a\int_0^b w_x(x,y)\cos\frac{m\pi\, x}{a}\sin\frac{n\pi\, y}{b}\,dx\,dy \quad \text{...............................} \quad (52)$$

$$g_{mn}=\frac{4}{ab}\int_0^a\int_0^b g(x,y)\cos\frac{m\pi\, x}{a}\sin\frac{n\pi\, y}{b}\,dx\,dy \quad \text{...............................} \quad (53)$$

$$w_{ymn}=\frac{4}{ab}\int_0^a\int_0^b w_y(x,y)\sin\frac{m\pi\, x}{a}\cos\frac{n\pi\, y}{b}\,dx\,dy \quad \text{...............................} \quad (54)$$

$$h_{mn}=\frac{4}{ab}\int_0^a\int_0^b h(x,y)\sin\frac{m\pi\, x}{a}\cos\frac{n\pi\, y}{b}\,dx\,dy \quad \text{..................................} \quad (55)$$

$$q_{mn}=\frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi\, x}{a}\sin\frac{n\pi\, y}{b}\,dx\,dy \quad \text{.................................} \quad (56)$$

Even though these are not a full representation of a Fourier series these functions will give exact values in the intervals $0 \le x \le a$ and $0 \le y \le b$.
By substituting Eq. 47 through 51 in Eq. 39 and differentiating when it is required yields:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Bigg[\left(\frac{m\pi}{a}\right)^3 g_{mn}+\nu\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)h_{mn}+\left(\frac{n\pi}{b}\right)^3 h_{mn}+\nu\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)^2 g_{mn}$$
$$+(1-\nu)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)^2 w_{xmn}+(1-\nu)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)w_{ymn}\Bigg]\sin\frac{m\pi\, x}{a}\sin\frac{n\pi\, y}{b}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}}{D}\sin\frac{m\pi\, x}{a}\sin\frac{n\pi\, y}{b}$$
$$\text{.............................} \quad (57)$$

Equating terms result in the following equation to be satisfied:

$$\left(\frac{m\pi}{a}\right)\left[\left(\frac{m\pi}{a}\right)^3 g_{mn} + \left(\frac{n\pi}{b}\right)^2 \left(\nu g_{mn} + (1-\nu) w_{xmn}\right)\right] + \left(\frac{n\pi}{b}\right)\left[\left(\frac{n\pi}{b}\right)^3 h_{mn} + \left(\frac{m\pi}{a}\right)^2 \left(\nu h_{mn} + (1-\nu) w_{ymn}\right)\right] = \frac{q_{mn}}{D}$$

…………………………………….. (58)

Equation 58 is to be equated term by term for every $m = 1, 2, 3, 4,\ldots.$ and $n = 1, 2, 3, 4,\ldots..$ Thus the solution can be achieved by selecting a set of coefficients p_{jk} for $j = 1, 2, 3, \ldots., r$ and $k = 1, 2, 3, \ldots., t$ and for an acceptable set of n and m then integrate Eq. 52 through 56 numerically with an acceptable accuracy and see if Eq. 58 is satisfied for every n and m. If it is not satisfied then updated p_{jk} with an acceptable conversion algorithm such as Newton Raphson Method with the Jacobian matrix. As we said before there is one deflection per load thus the coefficient p_{jk} must be unique, thus there is one root for the solution of Eq. 58. The initial vector p_{jk} can be taken by letting $g(x,y) = w_x$ and $h(x,y) = w_y$ so that the result gives the solution for small deflections as a start (see Timoshenko Eq 101, 102 pp81) this makes Eq. 58:

$$\left[\left(\frac{m\pi}{a}\right)^3 + \left(\frac{n\pi}{b}\right)^2\right]\left[\left(\frac{m\pi}{a}\right) w_{xmn} + \left(\frac{n\pi}{b}\right) w_{ymn}\right] = \frac{q_{mn}}{D} \quad \text{…………………….. (59)}$$

It is seen that this becomes a matrix inversion for p_{jk} if $jk = mn$ and the matrix to be inverted is an (mn) x (mn) inverted matrix. All of the coefficients of the left of Eq. 52 to 55 can be evaluated exactly without numerical integrations. The proposed general solution promises to be exact for an acceptable accuracy for the deflection. In the real world depending on the application there always a defined acceptable tolerance set by the engineer.

To control this tolerance it is best to represent Eq. 40 as follows:

$$\frac{D}{b^4} w = \sum_{j=0}^{r} \sum_{k=0}^{t} p_{jk} \left(\frac{x}{a}\right)^j \left(\frac{y}{b}\right)^k \quad \text{……………………………………………. (60)}$$

When $b > a$, this is done in order to have a more conversion series with a good representation of the deflection for $0 \le \frac{x}{a} \le 1$ and $0 \le \frac{y}{b} \le 1$.

Boundary Condition:
The boundary condition of the plate at the edges or internally can be achieved by selecting a conversion series for Eq. 60 that satisfies the boundary conditions for any set of *j* and *k*. This will become evident the following application examples.

Section 1:

A. *Clamped Rectangular Plate at the boundary edge*:

By inspection at $x = 0$ and $x = a$ Eq. 60 must have a root at $x = 0$ and $x = a$ to have zero deflections. Also for a clamped plate $w_x = 0$ at $x = 0$ and $x = a$ so Eq. 60 must have another root at $x = 0$ and $x = a$ to have zero slopes. This can also be said for the boundaries at *y* and Eq. 60 can be written as:

$$\frac{D}{b^4}w = \left(\frac{x}{a}\right)^2\left(\frac{x}{a}-1\right)^2\left(\frac{y}{b}\right)^2\left(\frac{y}{b}-1\right)^2\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left(\frac{x}{a}\right)^j\left(\frac{y}{b}\right)^k$$

$$=\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left[\left(\frac{x}{a}\right)^{j+4}-2\left(\frac{x}{a}\right)^{j+3}+\left(\frac{x}{a}\right)^{j+2}\right]\left[\left(\frac{y}{b}\right)^{k+4}-2\left(\frac{y}{b}\right)^{k+3}+\left(\frac{y}{b}\right)^{k+2}\right]$$

…………………….. (61)

Thus w_{xx} and w_{yy} can be written as:

$$\frac{a^2D}{b^4}w_{xx}=\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left[(j+4)(j+3)\left(\frac{x}{a}\right)^{j+2}-2(j+3)(j+2)\left(\frac{x}{a}\right)^{j+1}+(j+2)(j+1)\left(\frac{x}{a}\right)^{j}\right]$$
$$\times\left[\left(\frac{y}{b}\right)^{k+4}-2\left(\frac{y}{b}\right)^{k+3}+\left(\frac{y}{b}\right)^{k+2}\right]$$

…………………………… (62)

$$\frac{D}{b^2}w_{yy}=\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left[(k+4)(k+3)\left(\frac{y}{b}\right)^{k+2}-2(k+3)(k+2)\left(\frac{y}{b}\right)^{k+1}+(k+2)(k+1)\left(\frac{y}{b}\right)^{k}\right]$$
$$\times\left[\left(\frac{x}{a}\right)^{j+4}-2\left(\frac{x}{a}\right)^{j+3}+\left(\frac{x}{a}\right)^{j+2}\right]$$

……………………… (63)

Using Eq. 62 and 63 we can find the moments at boundary knowing the slopes are zero at the boundary such that $1/r_x = -w_{xx}$ at $x = 0$ etc. and the moments becomes:

$$\left.M_x\right|_{x=0} = -2\frac{b^4}{a^2}\sum_{k=0}^{t} p_{0k}\left[\left(\frac{y}{b}\right)^{k+4} - 2\left(\frac{y}{b}\right)^{k+3} + \left(\frac{y}{b}\right)^{k+2}\right]$$

$$\left.M_y\right|_{x=0} = \nu \left.M_x\right|_{x=0} \qquad \text{.......................... (64)}$$

$$\left.M_x\right|_{x=a} = -2\frac{b^4}{a^2}\sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left[\left(\frac{y}{b}\right)^{k+4} - 2\left(\frac{y}{b}\right)^{k+3} + \left(\frac{y}{b}\right)^{k+2}\right]$$

$$\left.M_y\right|_{x=a} = \nu \left.M_x\right|_{x=a}$$

$$\left.M_y\right|_{y=0} = -2b^2\sum_{j=0}^{r} p_{j0}\left[\left(\frac{x}{a}\right)^{j+4} - 2\left(\frac{x}{a}\right)^{j+3} + \left(\frac{x}{a}\right)^{j+2}\right]$$

$$\left.M_x\right|_{y=0} = \nu \left.M_y\right|_{y=0} \qquad \text{........................... (65)}$$

$$\left.M_y\right|_{y=b} = -2b^2\sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left[\left(\frac{x}{a}\right)^{j+4} - 2\left(\frac{x}{a}\right)^{j+3} + \left(\frac{x}{a}\right)^{j+2}\right]$$

$$\left.M_x\right|_{y=b} = \nu \left.M_y\right|_{x=b}$$

We can see from Eq. 64 and 65 that the moments on the boundaries are different functions with different coefficients p_{jk} and Eq. 61 is a representation of a clamped plate.

B. *Clamped Rectangular Plate at the boundary edge and symmetric load*:

For a symmetric load we first translate the deflection to the center of the plate and look at the deflection we have:

$$\frac{D}{b^4}\overline{w} = \sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left[\left(\frac{x}{a}\right)^2 - \frac{1}{4}\right]^{j+2}\left[\left(\frac{y}{b}\right)^2 - \frac{1}{4}\right]^{k+2} \qquad \text{.............................. (66)}$$

Where Taylor expansion is done on x^2 and y^2 to obtain symmetry and the roots at $x = \pm\frac{a}{2}$ and $y = \pm\frac{b}{2}$ satisfied the boundary conditions. When translating back to a corner of the plate Eq. 66 becomes:

$$\frac{D}{b^4}w = \sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j+2}\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k+2} \qquad \text{........................... (67)}$$

And the moments becomes:

$$M_x\Big|_{x=0} = -2\frac{b^4}{a^2}\sum_{k=0}^{t} p_{0k}\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k+2}$$

$$M_y\Big|_{x=0} = \nu M_x\Big|_{x=0}$$

$$M_x\Big|_{x=a} = -2\frac{b^4}{a^2}\sum_{k=0}^{t} p_{0k}\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k+2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (68)$$

$$M_y\Big|_{x=a} = \nu M_x\Big|_{x=a}$$

$$M_y\Big|_{y=0} = -2b^2\sum_{j=0}^{r} p_{j0}\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j+2}$$

$$M_x\Big|_{y=0} = \nu M_y\Big|_{y=0}$$

$$M_y\Big|_{y=b} = -2b^2\sum_{j=0}^{r} p_{j0}\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j+2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (69)$$

$$M_x\Big|_{y=b} = \nu M_y\Big|_{x=b}$$

And we show symmetry in the moments.

Now we seek the moments at $x = a/2$ and $y = b/2$. First we find w_{xx} and w_{yy} from Eq. 67 as follows:

$$w_{xx} = \frac{b^4}{Da^2}\sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left\{(j+2)(j+1)\left[2\left(\frac{x}{a}\right)-1\right]\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j} + 2(j+2)\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j+1}\right\} \times\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k+2}$$

$$w_{yy} = \frac{b^2}{D}\sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left\{(k+2)(k+1)\left[2\left(\frac{y}{b}\right)-1\right]\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k} + 2(k+2)\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k+1}\right\} \times\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j+2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (70)$$

Substitute $x = a/2$ and $y = b/2$ in Eq. 70 yields:

$$w_{xx}=\frac{2b^4}{Da^2}\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}(j+2)\left[-\frac{1}{4}\right]^{j+1}\left[-\frac{1}{4}\right]^{k+2}=-\frac{b^4}{2Da^2}\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}(j+2)\left[-\frac{1}{4}\right]^{j+k+1}$$

$$w_{yy}=\frac{2b^2}{D}\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}(k+2)\left[-\frac{1}{4}\right]^{j+2}\left[-\frac{1}{4}\right]^{k+1}=-\frac{b^2}{2D}\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}(k+2)\left[-\frac{1}{4}\right]^{j+k+1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(72)$$

Since $w_x = w_y = 0$ at $x = a/2$ and $y = b/2$ the moments at the center becomes:

$$M_x==\frac{b^2}{2}\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left[\frac{b^2}{a^2}(j+2)+\nu(k+2)\right]\left[-\frac{1}{4}\right]^{j+k+1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (73)$$

$$M_x==\frac{b^2}{2}\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left[(k+2)+\nu\frac{b^2}{a^2}(j+2)\right]\left[-\frac{1}{4}\right]^{j+k+1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74)$$

For a square plate $a = b$ then w is the same for the coordinate (x,y) and (y,x) then $j=k$ and we can write $p_{jk} = p_{kj} = p_{jj} = p_j$ and substitute in Eq. 67 yields:

$$\frac{D}{b^4}w=\sum_{j=0}^{r}\sum_{j=0}^{r}p_j\left[\left(\left(\frac{x}{a}\right)^2-\left(\frac{x}{a}\right)\right)\left(\left(\frac{y}{a}\right)^2-\left(\frac{y}{a}\right)\right)\right]^{2j+4}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (75)$$

C. <u>*Clamped Rectangular Plate with a Clamped ellipse Built in the center*</u>
See Fig. 1

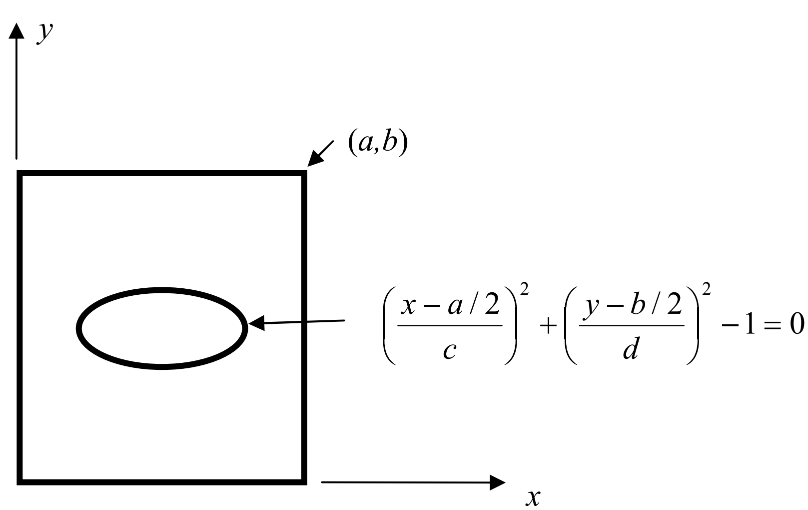

Fig. 1- Clamped Rectangular Plate with a Clamped ellipse Built in the center

By inspection at $x = 0$ and $x = a$ Eq. 60 must have a root at $x = 0$ and $x = a$ and at the contour of the ellipse to have zero deflections. Also for a clamped plate $w_x = 0$ at $x = 0$ and $x = a$ and at the contour of the ellipse so Eq. 60 must have another root at $x = 0$ and $x = a$ and at the contour of the ellipse to have zero slopes. This can also be said for the boundaries at y and Eq. 60 can be written as Eq. 61 with the ellipse contour:

$$
\begin{aligned}
\frac{D}{b^4}w &= \left(\frac{x}{a}\right)^2\left(\frac{x}{a}-1\right)^2\left(\frac{y}{b}\right)^2\left(\frac{y}{b}-1\right)^2\left[\left(\frac{a}{c}\right)^2\left(\frac{x}{a}-\frac{1}{2}\right)^2+\left(\frac{b}{d}\right)^2\left(\frac{y}{b}-\frac{1}{2}\right)^2-1\right]^2\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left(\frac{x}{a}\right)^j\left(\frac{y}{b}\right)^k \\
&= \sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left[\left(\frac{x}{a}\right)^{j+4}-2\left(\frac{x}{a}\right)^{j+3}+\left(\frac{x}{a}\right)^{j+2}\right]\left[\left(\frac{y}{b}\right)^{k+4}-2\left(\frac{y}{b}\right)^{k+3}+\left(\frac{y}{b}\right)^{k+2}\right] \\
&\qquad\times\left[\left(\frac{a}{c}\right)^2\left(\frac{x}{a}-\frac{1}{2}\right)^2+\left(\frac{b}{d}\right)^2\left(\frac{y}{b}-\frac{1}{2}\right)^2-1\right]^2 \\
&= \sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left[\left(\frac{x}{a}\right)^{j+4}-2\left(\frac{x}{a}\right)^{j+3}+\left(\frac{x}{a}\right)^{j+2}\right]\left[\left(\frac{y}{b}\right)^{k+4}-2\left(\frac{y}{b}\right)^{k+3}+\left(\frac{y}{b}\right)^{k+2}\right] \\
&\qquad\times\left[\left(\frac{a}{c}\right)^2\left(\left(\frac{x}{a}\right)^2-\frac{x}{a}+\frac{1}{4}\right)+\left(\frac{b}{d}\right)^2\left(\left(\frac{y}{b}\right)^2-\frac{y}{b}+\frac{1}{4}\right)\right]^2
\end{aligned}
\qquad\text{(76)}
$$

One can see Eq. 76 can be expanded to give another expression on Taylor polynomial and the solution can be obtained.

D. *Clamped Triangular Plate*:

Similar to what we have done before by inspection at $x = 0$ and $y = 0$ and at the edge of triangle Eq. 60 must have a root at $x = 0$ and $y = 0$ and at the edge of triangle (see Fig 2) to have zero deflections. Also for a clamped plate $w_x = 0$ at $x = 0$ and at the edge of triangle $w_y = 0$ at $y = 0$ and at the edge of triangle then Eq. 60 must have another root at $x = 0$ and $y = 0$ and at the edge of triangle to have zero slopes and Eq. 60 can be written as:

$$
\frac{D}{b^4}w = \left(\frac{x}{a}\right)^2\left(\frac{y}{b}\right)^2\left[\frac{x}{a}+\frac{y}{b}-1\right]^2\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left(\frac{x}{a}\right)^j\left(\frac{y}{b}\right)^k \qquad\text{(77)}
$$

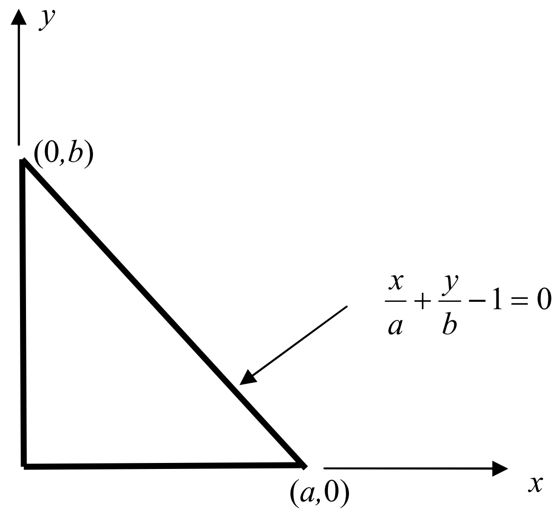

Fig. 2- Clamped Triangular Plate all sides

One can see Eq. 77 can be expanded to give another expression on Taylor polynomial and the solution can be obtained.

E. *Two Side Clamped Rectangular Plate with a Clamped ellipse Built at the edge*

Similar to what we have done before by inspection at $x = 0$ and $y = 0$ and at the edge of the ellipse Eq. 60 must have a root at $x = 0$ and $y = 0$ and at the edge of the ellipse (see Fig 3) to have zero deflections. Also for a clamped plate $w_x = 0$ at $x = 0$ and at the edge of the ellipse $w_y = 0$ at $y = 0$ and at the edge of the ellipse then Eq. 60 must have another root at $x = 0$ and $y = 0$ and at the edge of the ellipse to have zero slopes and Eq. 60 can be written as:

$$\frac{D}{b^4}w=\left(\frac{x}{a}\right)^2\left(\frac{y}{b}\right)^2\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-1\right]^2\sum_{j=0}^{r}\sum_{k=0}^{t}p_{jk}\left(\frac{x}{a}\right)^j\left(\frac{y}{b}\right)^k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (78)$$

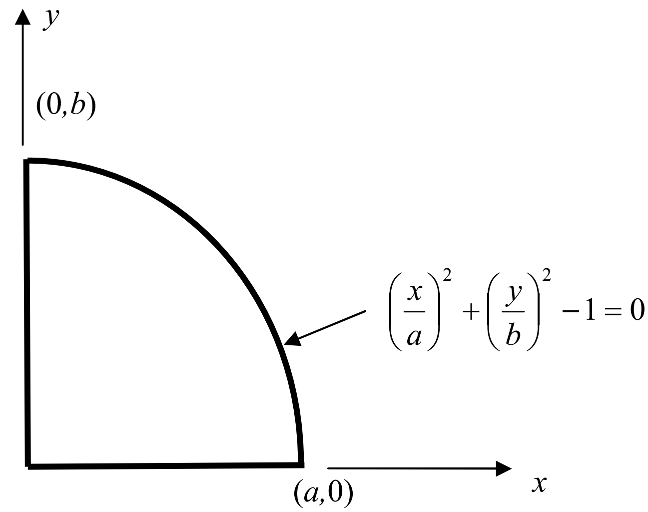

Fig. 3- Clamped Plate all sides

One can see Eq. 78 can be expanded to give another expression on Taylor polynomial and the solution can be obtained.

F. *Two Side Clamped Rectangular Plate with a Clamped Polynomial Curve Built at the edge*

Similar to what we have done before by inspection at $x = 0$ and $y = 0$ and at the edge of the curve Eq. 60 must have a root at $x = 0$ and $y = 0$ and at the edge of the curve (see Fig 4) to have zero deflections. Also for a clamped plate $w_x = 0$ at $x = 0$ and at the edge of the curve $w_y = 0$ at $y = 0$ and at the edge of the curve then Eq. 60 must have another root at $x = 0$ and $y = 0$ and at the edge of the curve to have zero slopes and Eq. 60 can be written as:

$$\frac{D}{b^4} w = \left(\frac{x}{a}\right)^2 \left(\frac{y}{b}\right)^2 [R(x,y)]^2 \sum_{j=0}^{r} \sum_{k=0}^{t} p_{jk} \left(\frac{x}{a}\right)^j \left(\frac{y}{b}\right)^k \quad \text{.......................................} (79)$$

Where the curve is represented as polynomial:

$$R(x,y) = \sum_{j'=0}^{r'} \sum_{k'=0}^{t'} \bar{p}_{j'k'} x^{j'} y^{k'} \quad \text{...} (80)$$

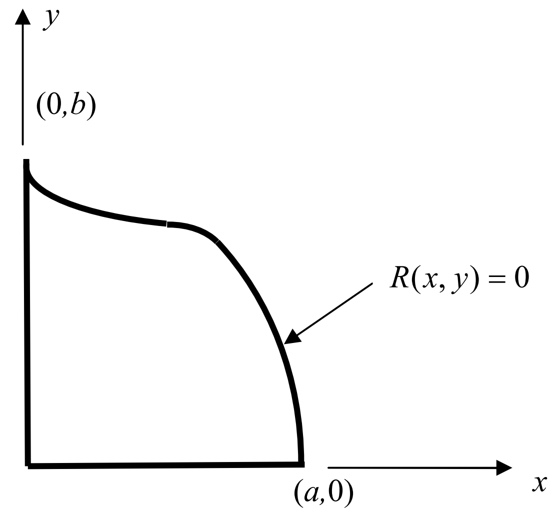

Fig. 4- Clamped Plate all sides

One can see Eq. 79 and Eq. 80 can be expanded to give another expression on Taylor polynomial and the solution can be obtained.

G. *Discussion on Simple span and Clamped Plates with Large Deflection:*

Clamped plates are basically simple supported plate with a specific moment at the edge that make the slopes at the edge flat or zero. If there is large deflection on such a plate where the plate is unattached to its support, and the plate a little oversized over the support, then large deflection will cause the plate to deflect enough at the edges of the plate to loose its support at the edge, where the friction at the support due to the vertical load is ignored, (assumed smooth). This can happen when the bending stresses in the plate did not reach failure, which is the case in most thin plates, see Fig. 5 for a rectangular plate. Because the plate is clamped this problem of loosing support at the edge is not likely to happen in reality but it is more likely for an unattached simple supported plates where the friction at support due to the vertical load is ignored, assumed smooth. (This can happen in addition to reverse deflection vertically in section of the support, see Section 3 for dealing with this problem) However, the above solution offers to solve various clamped plate problems that has not been solved before and if large deflection is not an issue then the initial value of Eq. 59 is sufficient for solution which requires only one matrix inversion.

We give the equations for loosing supports for a symmetric loading on a rectangular plate with any edge condition as follows:

$$\Delta_x \cong \frac{1}{2}\left[\int_0^a \frac{dx}{\sqrt{1+\left[w_x(x,b/2)\right]^2}} - a\right]$$
$$\Delta_y \cong \frac{1}{2}\left[\int_0^b \frac{dy}{\sqrt{1+\left[w_y(a/2,y)\right]^2}} - b\right] \quad \ldots\ldots\ldots (81)$$

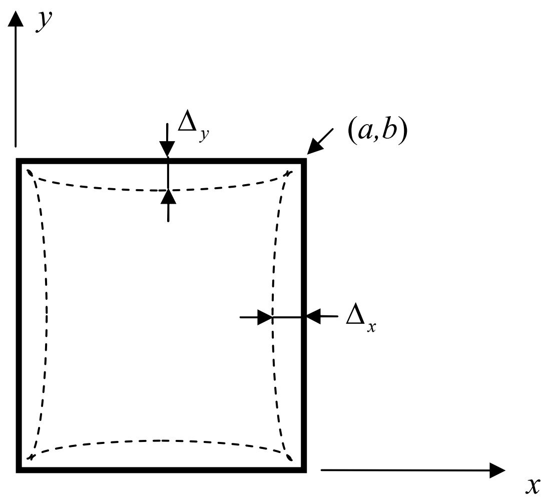

Fig. 5 – loss of support in a plate

Section 2:

A. *Rectangular Simply Supported Plate:*

By inspection at $x = 0$ and $x = a$ Eq. 60 must have a root at $x = 0$ and $x = a$ to have zero deflections. This can also be said for the boundaries at y and Eq. 60 can be written as:

$$\frac{D}{b^4} w = \left(\frac{x}{a}\right)\left(\frac{x}{a}-1\right)\left(\frac{y}{b}\right)\left(\frac{y}{b}-1\right)\sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left(\frac{x}{a}\right)^j\left(\frac{y}{b}\right)^k$$
$$= \sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left[\left(\frac{x}{a}\right)^{j+2} - \left(\frac{x}{a}\right)^{j+1}\right]\left[\left(\frac{y}{b}\right)^{k+2} - \left(\frac{y}{b}\right)^{k+1}\right]$$

……………………. (82)

Thus w_x and w_y can be written as:

$$\frac{aD}{b^4} w_x = \sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left[(j+2)\left(\frac{x}{a}\right)^{j+1} - (j+1)\left(\frac{x}{a}\right)^{j}\right]\left[\left(\frac{y}{b}\right)^{k+2} - \left(\frac{y}{b}\right)^{k+1}\right]$$

……………………… (83)

$$\frac{D}{b^3} w_y = \sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left[(k+2)\left(\frac{y}{b}\right)^{k+1} - (k+1)\left(\frac{y}{b}\right)^{k}\right]\left[\left(\frac{x}{a}\right)^{j+2} - \left(\frac{x}{a}\right)^{j+1}\right]$$

…………………….….. (84)

Thus w_{xx} and w_{yy} can be written as:

$$\frac{a^2 D}{b^4} w_{xx} = \sum_{k=0}^{t}\left\{2p_{0k} + \sum_{j=1}^{r} p_{jk}\left[(j+2)(j+1)\left(\frac{x}{a}\right)^{j+2} - (j+1)j\left(\frac{x}{a}\right)^{j-1}\right]\right\}\left[\left(\frac{y}{b}\right)^{k+2} - \left(\frac{y}{b}\right)^{k+1}\right]$$

………………………… (85)

$$\frac{D}{b^2} w_{yy} = \sum_{j=0}^{r}\left\{2p_{k0} + \sum_{k=1}^{t} p_{jk}\left[(k+2)(k+1)\left(\frac{y}{b}\right)^{j+2} - (k+1)k\left(\frac{y}{b}\right)^{k-1}\right]\right\}\left[\left(\frac{x}{a}\right)^{j+2} - \left(\frac{x}{a}\right)^{j+1}\right]$$

…………………….….. (86)

Using the equation for finding the moments at the edges we write:

$$M_x\Big|_{\substack{x=0\\x=a}} = -D\frac{w_{xx}}{\left[1+(w_x)^2\right]^{\frac{3}{2}}} - \nu D\frac{w_{yy}}{\left[1+(w_y)^2\right]^{\frac{3}{2}}} = 0$$

$$M_y\Big|_{\substack{x=0\\x=a}} = -D\frac{w_{yy}}{\left[1+(w_{yy})^2\right]^{\frac{3}{2}}} - \nu D\frac{w_{xx}}{\left[1+(w_{xx})^2\right]^{\frac{3}{2}}} = 0$$

………………………………… (87)

When solving the two equation for $1/r_x = 0$ and $1/r_y = 0$ at $x = 0$ and $x = a$, we see the only way Eq. 87 is satisfied is when

$$w_{xx}\Big|_{\substack{x=0\\x=a}} = 0 \quad \text{and} \quad w_{yy}\Big|_{\substack{x=0\\x=a}} = 0$$ ……………………………………………………. (88)

Similarly the argument holds for $y = 0$ and $y = b$, so

$$w_{xx}\Big|_{\substack{y=0\\y=b}} = 0 \quad \text{and} \quad w_{yy}\Big|_{\substack{y=0\\y=b}} = 0$$ ……………………………………………………. (89)

We can see that is true already by substitution in Eq. 85 and Eq. 86 that:

$$w_{xx}\Big|_{\substack{y=0\\y=b}} = 0 \quad \text{and} \quad w_{yy}\Big|_{\substack{x=0\\x=a}} = 0$$

When using Eq. 85 we find

$$0 = \sum_{k=0}^{t}\{2p_{0k} - 2p_{1k}\}\left[\left(\frac{y}{b}\right)^{k+2} - \left(\frac{y}{b}\right)^{k+1}\right] \quad \text{at} \quad x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (90)$$

Thus set $p_{0k} = p_{1k}$ in the polynomial of Eq. 82 and the condition is satisfied.

$$0 = \sum_{k=0}^{t}\left\{2p_{0k} + \sum_{j=1}^{r} p_{jk}\,2(j+1)\right\}\left[\left(\frac{y}{b}\right)^{k+2} - \left(\frac{y}{b}\right)^{k+1}\right] \quad \text{at} \quad x = a$$

$$0 = \sum_{k=0}^{t}\left\{6p_{0k} + \sum_{j=2}^{r} p_{jk}\,2(j+1)\right\}\left[\left(\frac{y}{b}\right)^{k+2} - \left(\frac{y}{b}\right)^{k+1}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (91)$$

Or set

$$p_{0k} = p_{1k} = -\frac{1}{3}\sum_{j=2}^{r} p_{jk}\,2(j+1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (92)$$

Similarly for $y = 0$ and $y = b$ we have the condition:

$$p_{j0} = p_{j1} = -\frac{1}{3}\sum_{k=2}^{t} p_{jk}\,2(k+1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (93)$$

Thus when selecting these coefficients of Eq. 92 and 93 and substituting in Eq. 82 the boundary condition are satisfied.

B. *Simple Suported Rectangular Plate at the boundary edge and symmetric load*:

For a symmetric load we first translate the deflection to the center of the plate and look at the deflection we have:

$$\frac{D}{b^4}\overline{w} = \sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left[\left(\frac{x}{a}\right)^2 - \frac{1}{4}\right]^{j+1}\left[\left(\frac{y}{b}\right)^2 - \frac{1}{4}\right]^{k+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (94)$$

Where Taylor expansion is done on x^2 and y^2 to obtain symmetry and the roots at $x = \pm\frac{a}{2}$ and $y = \pm\frac{b}{2}$ satisfied the boundary conditions. When translating back to a corner of the plate Eq. 94 becomes:

$$\frac{D}{b^4}w = \sum_{j=0}^{r}\sum_{k=0}^{t} p_{jk}\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j+1}\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k+1} \qquad (95)$$

And w_x and w_y becomes:

$$\frac{aD}{b^4}w_x = \sum_{k=0}^{t}\left\{p_{0k}\left[2\left(\frac{x}{a}\right)-1\right] + 2p_{1k}\left[2\left(\frac{x}{a}\right)^3 - 3\left(\frac{x}{a}\right)^2 + \left(\frac{x}{a}\right)\right] + \sum_{j=2}^{r}(j+1)p_{jk}\left[2\left(\frac{x}{a}\right)-1\right]\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j}\right\} \times\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k+1} \qquad (96)$$

$$\frac{D}{b^3}w_y = \sum_{j=0}^{r}\left\{p_{j0}\left[2\left(\frac{y}{b}\right)-1\right] + 2p_{j1}\left[2\left(\frac{y}{b}\right)^3 - 3\left(\frac{y}{b}\right)^2 + \left(\frac{y}{b}\right)\right] + \sum_{k=2}^{t}(k+1)p_{jk}\left[2\left(\frac{y}{b}\right)-1\right]\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k}\right\} \times\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j+1} \qquad (97)$$

And w_{xx} and w_{yy} becomes:

$$\frac{a^2D}{b^4}w_{xx} = \sum_{k=0}^{t}\left\{2p_{0k} + 2p_{1k}\left[6\left(\frac{x}{a}\right)^2 - 6\left(\frac{x}{a}\right) + 1\right] + \sum_{j=2}^{r}(j+1)jp_{jk}\left[2\left(\frac{x}{a}\right)-1\right]\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j-1}\right\} \times\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k+1} \qquad (98)$$

$$\frac{D}{b^2}w_{yy} = \sum_{j=0}^{r}\left\{2p_{j0} + 2p_{j1}\left[6\left(\frac{y}{b}\right)^2 - 6\left(\frac{y}{b}\right)^1 + 1\right] + \sum_{k=2}^{t}(k+1)kp_{jk}\left[2\left(\frac{y}{b}\right)-1\right]\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]^{k-1}\right\} \times\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^{j+1} \qquad (99)$$

As before we require $w_{xx}\big|_{\substack{x=0\\x=a}} = 0$ and $w_{yy}\big|_{\substack{y=0\\y=b}} = 0$. From Eq. 98 and Eq. 99 we must have:

$p_{j0} = - p_{j1}$ and $p_{0k} = - p_{1k}$ ………………………………………… (100)

Thus when selecting these coefficients and substituting in Eq. 94 the boundary condition are satisfied.

Now we seek the moments at $x = a/2$ and $y = b/2$. First we find w_{xx} and w_{yy} from Eq. 98 and Eq. 99 and using Eq. 100 yields:

$$\begin{aligned} w_{xx} &= \frac{6b^4}{a^2 D}\sum_{k=0}^{t} p_{0k}\left[-\frac{1}{4}\right]^{k+1} \\ w_{yy} &= \frac{6b^2}{D}\sum_{j=0}^{r} p_{j0}\left[-\frac{1}{4}\right]^{j+1} \end{aligned} \quad \text{……………………………………………… (101)}$$

Since $w_x = w_y = 0$ at $x = a/2$ and $y = b/2$ the moments at the center becomes:

$$\begin{aligned} M_x &= -\frac{6b^4}{a^2}\sum_{k=0}^{t} p_{0k}\left[-\frac{1}{4}\right]^{k+1} - 6\nu b^2\sum_{j=0}^{r} p_{j0}\left[-\frac{1}{4}\right]^{j+1} \\ M_y &= -6b^2\sum_{j=0}^{r} p_{j0}\left[-\frac{1}{4}\right]^{j+1} - \frac{6\nu b^4}{a^2}\sum_{k=0}^{t} p_{0k}\left[-\frac{1}{4}\right]^{k+1} \end{aligned} \quad \text{………………………….. (102)}$$

For a square plate $a = b$ then w is the same for the coordinate (x,y) and (y,x) then $j=k$ and we can write $p_{jk} = p_{kj} = p_{jj} = p_j$ and substituting in Eq. 95 yields:

$$\frac{D}{b^4} w = \sum_{j=0}^{r}\sum_{j=0}^{r} p_j\left[\left(\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right)\left(\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right)\right]^{2j+2} \quad \text{………………………… (103)}$$

Many other boundary conditions can be satisfied with finding the proper Taylor series expansion. One more conditions will be addressed for the case of a rectangular plate with one free edge and the others are clamped edge.

Section 3:

A. *Three Sided fixed rectangular Plate and one free edge:*

Consider Fig 6.

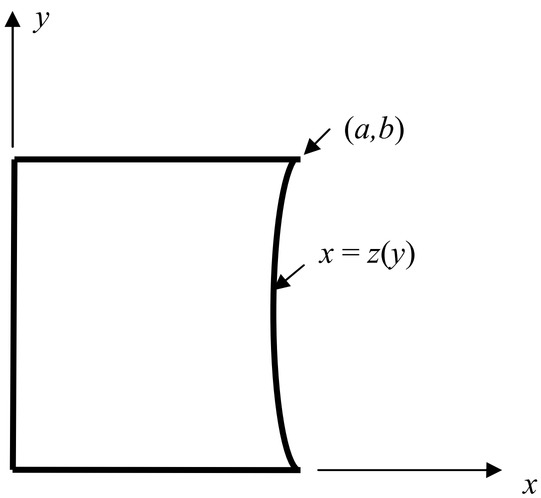

Fig. 6 Three sides clamped one side free

The function $x = z(y)$ is the final deflection of the free edge and can be expressed with an accurate approximation by Taylor polynomial as follows:

$$z(y) = \sum_{i=0}^{L} A_i \left(\frac{y}{b}\right)^i \qquad (104)$$

And as before the deflection can be written as

$$\frac{D}{b^4} w = \left(\frac{x}{a}\right)^2 \left(\frac{x}{a} - 1\right)^2 \left(\frac{y}{b}\right)^2 \sum_{j=0}^{r} \sum_{k=0}^{t} p_{jk} \left(\frac{x}{a}\right)^j \left(\frac{y}{b}\right)^k \qquad (105)$$

We enforce the boundary condition of the moments:

$$M_x\Big|_{x=z(y)} = -D\frac{w_{xx}}{\left[1+(w_x)^2\right]^{\frac{3}{2}}} - \nu D\frac{w_{yy}}{\left[1+(w_y)^2\right]^{\frac{3}{2}}} = 0$$

$$M_y\Big|_{x=z(y)} = -D\frac{w_{yy}}{\left[1+(w_{yy})^2\right]^{\frac{3}{2}}} - \nu D\frac{w_{xx}}{\left[1+(w_{xx})^2\right]^{\frac{3}{2}}} = 0 \quad \text{.................................} (106)$$

And the length of the plate at any y as:

$$\int_0^{z(y)} \frac{dx}{\sqrt{1+(w_x(x,y))^2}} = a \quad \text{...} (107)$$

Which makes the solution complicated, but the solution can be enforced a certain coordinates by taking a set of known y_i say (for example dividing b to increments b/T and $y_i = ib/T$, i = 0, 1,2, ..) then each Eq 106 and 107 is a set of i equations to add more equations to Eq. 58 and the algorithm of finding p_{jk} also finds A_i. Once A_i is found Eq. 106 and 107 can be used to verify the accuracy of other points beside y_i.

Non-Prismatic Plates

If we have a set of square plates with different thickness welded or attached together to make one big plate (where the perimeter can have triangular plates if needed) then we need to match the slope and deflection around each plate. This can be done by matching a set of points x_i and y_i for the function $x = z_1(y)$, $y = z_2(x)$ at the perimeter similar to the last example. Even though this starts to look like finite element it is a much better representation to include large deflection and more accurate.

General Solution using Point Loads and Moments:

Before giving the general solution with point loads and a moment at the point of application we will review the above solution using a one dimensional beam simply supported with no large deflection.

Let us use a set of Taylor polynomial to approximate the analysis for a simply supported beam with any loading see Fig. 7

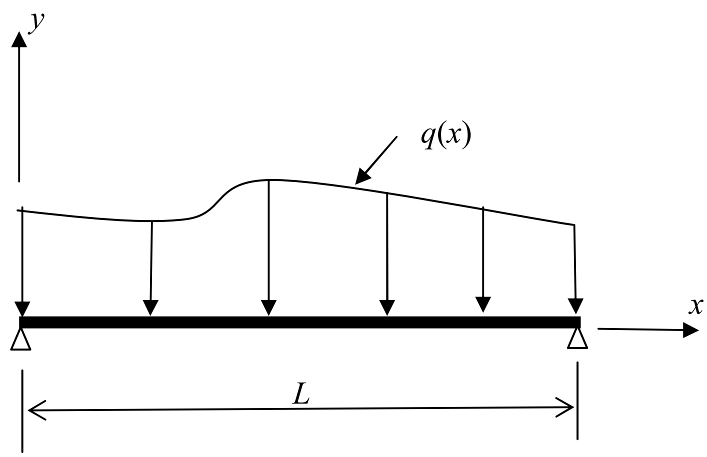

FIG 7. Simple Span Beam

Expressing the load $q(x)$ in Fourier Series yields,

$$q(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (108)$$

Now we pick Taylor series $y(x)$ that satisfies the boundary condition. Thus the series must have a root at $x = 0$ and $x = L$ and
$M(x)\big|_{x=0} = \ddot{y}\big|_{x=0} = 0$ and $M(x)\big|_{x=L} = \ddot{y}\big|_{x=L} = 0$

We will show the behavior of the solution for a 5th , 6th , 7th , 8th and 9th order polynomial and compare the results. We find the following equation satisfies the boundary condition:

5th order:

$$y(x) = \frac{L^4}{EI}\left[k_1\left(\frac{x}{L}\right)^5 - \left(\frac{5}{3}k_1 + \frac{k_3}{2}\right)\left(\frac{x}{L}\right)^4 + k_3\left(\frac{x}{L}\right)^3 + \left(\frac{2}{3}k_1 - \frac{k_3}{2}\right)\left(\frac{x}{L}\right)\right] \ldots\ldots\ldots\ldots (109)$$

6th order:

$$y(x) = \frac{L^4}{EI}\left[k_1\left(\frac{x}{L}\right)^6 + k_2\left(\frac{x}{L}\right)^5 - \left(\frac{5}{2}k_1 + \frac{5}{3}k_2 + \frac{k_4}{2}\right)\left(\frac{x}{L}\right)^4 + k_4\left(\frac{x}{L}\right)^3 + \left(\frac{3}{2}k_1 + \frac{2}{3}k_2 - \frac{k_4}{2}\right)\left(\frac{x}{L}\right)\right]$$

$$\ldots\ldots\ldots\ldots (110)$$

7th order:

$$y(x) = \frac{L^4}{EI}\left[k_1\left(\frac{x}{L}\right)^7 + k_2\left(\frac{x}{L}\right)^6 + k_3\left(\frac{x}{L}\right)^5 - \left(\frac{7}{2}k_1 + \frac{5}{2}k_2 + \frac{5}{3}k_3 + \frac{k_5}{2}\right)\left(\frac{x}{L}\right)^4 + k_5\left(\frac{x}{L}\right)^3 + \left(\frac{5}{2}k_1 + \frac{3}{2}k_2 + \frac{2}{3}k_3 - \frac{k_5}{2}\right)\left(\frac{x}{L}\right)\right] \quad \text{.......... (111)}$$

8th order:

$$y(x) = \frac{L^4}{EI}\left[k_1\left(\frac{x}{L}\right)^8 + k_2\left(\frac{x}{L}\right)^7 + k_3\left(\frac{x}{L}\right)^6 + k_4\left(\frac{x}{L}\right)^5 - \left(\frac{14}{3}k_1 + \frac{7}{2}k_2 + \frac{5}{2}k_3 + \frac{5}{3}k_4 + \frac{k_6}{2}\right)\left(\frac{x}{L}\right)^4 + k_6\left(\frac{x}{L}\right)^3 + \left(\frac{11}{3}k_1 + \frac{5}{2}k_2 + \frac{3}{2}k_3 + \frac{2}{3}k_4 - \frac{k_6}{2}\right)\left(\frac{x}{L}\right)\right]$$

.... (112)

9th order:

$$y(x) = \frac{L^4}{EI}\left[k_1\left(\frac{x}{L}\right)^9 + k_2\left(\frac{x}{L}\right)^8 + k_3\left(\frac{x}{L}\right)^7 + k_4\left(\frac{x}{L}\right)^6 + k_5\left(\frac{x}{L}\right)^5 - \left(6k_1 + \frac{14}{3}k_2 + \frac{7}{2}k_3 + \frac{5}{2}k_4 + \frac{5}{3}k_5 + \frac{k_7}{2}\right)\left(\frac{x}{L}\right)^4 + k_7\left(\frac{x}{L}\right)^3 + \left(5k_1 + \frac{11}{3}k_2 + \frac{5}{2}k_3 + \frac{3}{2}k_4 + \frac{2}{3}k_5 - \frac{k_7}{2}\right)\left(\frac{x}{L}\right)\right]$$

.... (113)

Where $k_1, k_2, k_3, k_4, k_5, k_6$ and k_7 are constants to be found. Now following the proposed solution we express the equation for the slope in Fourier series as:

$$\dot{y} = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L}$$

where .... (114)

$$b_n = \frac{2}{L}\int_0^L y(x)\cos\frac{n\pi x}{L}dx = 2\int_0^1 y(Lu)\cos n\pi u\, du$$

When integrating Eq. 109, 110, 111, 112 and 113 using Eq. 114 then differentiating three time to get the pressure and equating to Eq. 108 we obtain a set linear system of equations in k_i. When inverting the matrix for each polynomial equation we obtain the following solutions:

5th order:

$$\begin{bmatrix} k_1 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 & -0.052359878 \\ -0.130899694 & -0.087266463 \end{bmatrix} \begin{bmatrix} a_1/2 \\ a_2/2 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (115)$$

6th order

$$\begin{bmatrix} k_1 \\ k_2 \\ k_4 \end{bmatrix} = \begin{bmatrix} -0.024223654 & 0 & 0.072670961 \\ 0.072670961 & -0.052359878 & -0.218012883 \\ -0.104755807 & -0.087266463 & -0.078431662 \end{bmatrix} \begin{bmatrix} a_1/2 \\ a_2/2 \\ a_3/2 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (116)$$

7th order

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_5 \end{bmatrix} = \begin{bmatrix} -5.0307E-17 & 0.06562175 & -5.55112E-17 & -0.131243499 \\ -0.024223654 & -0.229676124 & 0.072670961 & 0.459352247 \\ 0.072670961 & 0.247129416 & -0.218012883 & -0.598978587 \\ -0.104755807 & -0.047469887 & -0.078431662 & -0.079593151 \end{bmatrix} \begin{bmatrix} a_1/2 \\ a_2/2 \\ a_3/2 \\ a_4/2 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (117)$$

8th order

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_6 \end{bmatrix} = \begin{bmatrix} 0.004447133 & 0 & -0.180108893 & 1.249E-16 & 0.277945823 \\ -0.017788533 & 0.06562175 & 0.720435572 & -0.131243499 & -1.11178329 \\ 0.003812983 & -0.229676124 & -1.062812804 & 0.459352247 & 1.752289761 \\ 0.050820917 & 0.247129416 & 0.66691391 & -0.598978587 & -1.365627767 \\ -0.106136585 & -0.047469887 & -0.022510129 & -0.079593151 & -0.086298663 \end{bmatrix} \begin{bmatrix} a_1/2 \\ a_2/2 \\ a_3/2 \\ a_4/2 \\ a_5/2 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (118)$$

9th order

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ K_5 \\ k_7 \end{bmatrix} = \begin{bmatrix} 2.7452E-16 & -0.040478794 & -5.35683E-15 & 0.518128567 & 9.13158E-15 & -0.655756467 \\ 0.004447133 & 0.182154574 & -0.180108893 & -2.331578551 & 0.277945823 & 2.950904103 \\ -0.017788533 & -0.269531601 & 0.720435572 & 4.158719394 & -1.11178329 & -5.429484287 \\ 0.003812983 & 0.093305925 & -1.062812804 & -3.674817977 & 1.752289761 & 5.23230919 \\ 0.050820917 & 0.074523349 & 0.66691391 & 1.610379068 & -1.365627767 & -2.796218283 \\ -0.106136585 & -0.053491716 & -0.022510129 & -0.002513738 & -0.086298663 & -0.097553632 \end{bmatrix} \begin{bmatrix} a_1/2 \\ a_2/2 \\ a_3/2 \\ a_4/2 \\ a_5/2 \\ a_6/2 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (119)$$

Now we investigate several loading condition:

1- Expressing the loading of Fig. 8 in Fourier Series yields,

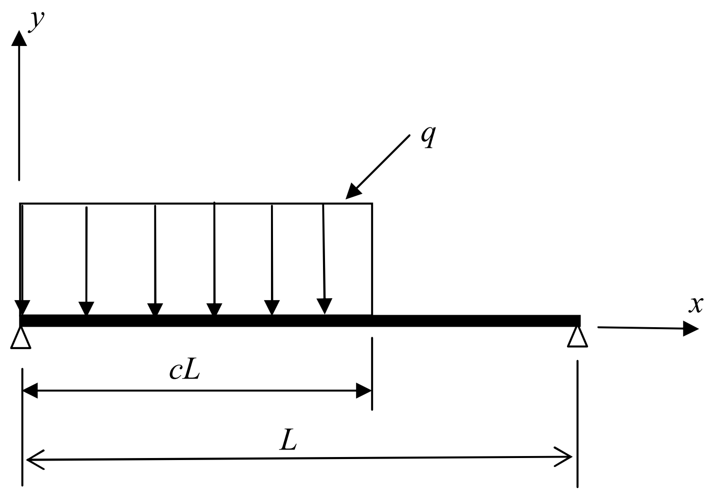

FIG 8. Partial Load on a Simple Span Beam

$$q(x) = 2q\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi c}{L}}{n\pi}\sin\frac{n\pi x}{L} \quad \ldots\ldots (120)$$

Thus:

$$\frac{a_n}{2} = \frac{1-\cos\frac{n\pi c}{L}}{n\pi} \quad \ldots\ldots (121)$$

Substituting in Eq. 115, 116 117, 118 and Eq. 119, and graphing the results for $c = .7$ yields Fig 11A and Fig 11B; Table 1 shows the results in comparing with the exact solution.

2- Expressing the loading of Fig. 9 in Fourier Series to obtain the pressure for a point load yields,

$$q(x) = -4q\sum_{i=1}^{\infty}\frac{\sin\frac{n\pi c}{L}\sin\frac{n\pi \varepsilon}{L}}{n\pi}\sin\frac{n\pi x}{L} \quad \ldots\ldots (122)$$

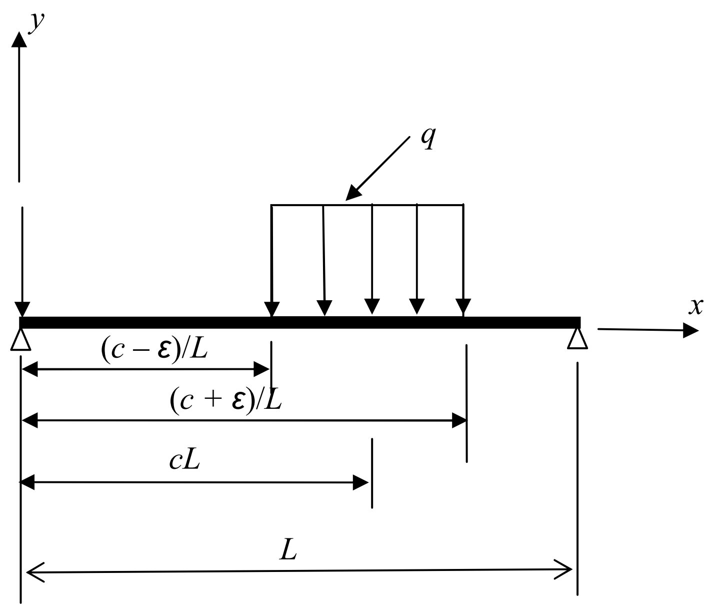

FIG 9. Load on Simple Span Beam to be converted to Point Load

Let the point load $P = 2\varepsilon\, q$ which is the load under q and substitute for q in Eq. 122
Thus

$$q(x) = -2\frac{P}{L}\sum_{i=1}^{\infty}\sin\frac{n\pi\, c}{L}\frac{\sin\frac{n\pi\,\varepsilon}{L}}{\frac{n\pi\,\varepsilon}{L}}\sin\frac{n\pi\, x}{L} \quad \text{……………………………..….. (123)}$$

Let ε go to zero and we get the pressure representing a point load as follows:

$$q(x) = -2\frac{P}{L}\sum_{i=1}^{\infty}\sin\frac{n\pi\, c}{L}\sin\frac{n\pi\, x}{L} \quad \text{……………………………………….. (124)}$$

Thus:

$$\frac{a_n}{2} = -\frac{P}{L}\sin\frac{n\pi\, c}{L} \quad \text{………………………………………………………… (125)}$$

Substituting in Eq. 115, 116 117, 118 and Eq. 119, and graphing the results for $c = .3$ yields Fig 12A and Fig 12B; Table 2 shows the results in comparing with the exact solution.

3- Expressing the loading of Fig. 10 in Fourier Series to obtain the pressure for a moment yields,

$$q(x) = 4q\sum_{i=1}^{\infty} \cos\frac{n\pi\, c}{L}\frac{\left[1-\cos\frac{n\pi\,\varepsilon}{L}\right]}{n\pi}\sin\frac{n\pi\, x}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (126)$$

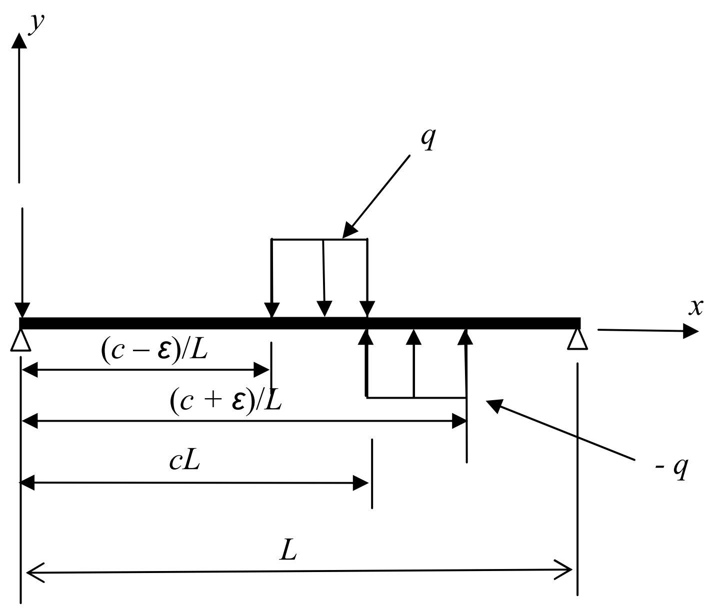

FIG 10. Load on Simple Span Beam to be converted to a Moment

Let the moment $M = 0.5\varepsilon\ (\varepsilon\ q) + 0.5\varepsilon\ (\varepsilon\ q) = q\varepsilon^2$ which is the moment for loading in Fig. 10, and substitute for q in Eq. 126

Thus

$$q(x) = 4\frac{M}{L^2}\sum_{i=1}^{\infty} n\pi\cos\frac{n\pi\, c}{L}\frac{\left[1-\cos\frac{n\pi\,\varepsilon}{L}\right]}{\left(\frac{n\pi\,\varepsilon}{L}\right)^2}\sin\frac{n\pi\, x}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (127)$$

Let ε go to zero and we get the pressure representing a Moment as follows:

$$q(x) = 2\frac{M}{L^2}\sum_{i=1}^{\infty} n\pi \cos\frac{n\pi\, c}{L} \sin\frac{n\pi\, x}{L} \quad \text{………………………………………….. (128)}$$

Thus:

$$\frac{a_n}{2} = -\frac{M}{L^2} n\pi \cos\frac{n\pi\, c}{L} \quad \text{………………………………………..…………….. (129)}$$

Substituting in Eq. 115, 116 117, 118 and Eq. 119, and graphing the results for c = .5 yields Fig 13A and Fig 13B; Table 3 shows the results in comparing with the exact solution.

All of the results shows Taylor approximation follows Fourier approximation and the error improves with adding higher term to Taylor polynomial.

For the point load we can see that under the pressure $\left.\frac{dV}{dx}\right|_{x=cL} \Rightarrow \infty$ so the pressure approaches infinity under the load. This has happen because we approximated the actual shear, which is a discontinuous function, by a continuous function. As in the actual shear curve at that point the shear is never defined under the load[2] since it has two values and the question becomes which value can we use. In practice the shear is always taken as $\max(|V(cL-\delta)|,|V(cL+\delta)|)$ for an appropriate increment δ where δ can be found by testing for ultimate values around cL. Thus the solution for the shear is correct for all values except under the load and can be taken as $\max(|V(cL-\delta)|,|V(cL+\delta)|)$ of the approximate Taylor polynomial. If we have a load that has been approximated with many point loads then it is best to determine what shear value to use under the load based on design practice and represent Taylor polynomial approximation as a discontinuous function with point values under the load. If we differentiate the shear to obtain the pressure then the pressure under the load can not be determined using Taylor polynomial as a continuous function and should be taken $P = V(cL-\delta) - V(cL+\delta)$.

[2] The reason the actual shear diagram has discontinuity because it is telling us in real life there no such thing as a point load and in reality it is some kind of a pressure with some small ε around the load as in Fig 9.

Similarly from the moment diagram we can see that under the shear $\left.\frac{dM}{dx}\right|_{x=cL} \Rightarrow \infty$ so the shear approaches infinity under the load. This has happen because we approximated the actual moment function, which is a discontinuous function, by a continuous function. As in the actual moment diagram at the point of application the moment is never defined under the moment[3] since it has two values and the question becomes which value can we use. In design practice the two moment is always taken as M^+ and M^- or $M(cL-\delta)$ and $M(cL+\delta)$ with their corresponding sign for an appropriate increment δ where δ can be found by testing for ultimate values around *cL*. Thus the solution for the moment is correct for all values except under the point of application and can be taken as $M(cL-\delta)$ and $M(cL+\delta)$ of the approximate Taylor polynomial. Thus it is best to represent Taylor polynomial approximation as a discontinuous function with point values under the moment. If we differentiate the moment to obtain the shear then the shear under the load can not be determined using Taylor polynomial as a continuous function and should be taken $V = \frac{1}{L}\left[M(cL-\delta) - M(cL+\delta)\right]$.

Finally if we differentiate the shear to obtain the pressure then the pressure under the load can not be determined using Taylor polynomial as a continuous function and should be taken zero. This can also be seen when using a slighted slanted line instead of a vertical line at the point of application, *cL*, in the moment diagram then differentiating twice to get the pressure resulting in a zero pressure.

[3] The reason the actual moment diagram has discontinuity because it is saying in real life there no such thing as a moment at a point of application and in reality it is some kind of a pressure with some small ε around the load as in Fig 10. For example if we try to put a moment using a pinion of a motor then in reality the pinion of the motor could never have a zero radius and the radius can only be as small as ε and transferring the load can only be possible by introducing some kind of a axis-symmetric pressure at the point of application from $-\varepsilon \le x - cL \le \varepsilon$.

Fifth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error in graph	M+ max Error
1	0.000%	0.000%	0.000%	0.000%
0.9	0.460%	-0.065%	13.186%	-0.456%
0.8	1.014%	-0.226%	19.070%	-1.466%
0.7	0.997%	-0.362%	13.237%	-2.413%
0.6	0.908%	-0.380%	8.817%	-2.733%
0.5	-0.206%	-0.121%	-18.672%	-2.778%
0.4	-2.342%	0.546%	-55.590%	-3.124%
0.3	-3.906%	1.386%	-86.714%	-4.667%
0.2	-6.173%	2.110%	-113.532%	-7.982%
0.1	-11.445%	2.694%	-131.784%	-13.191%

Sixth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max Error	M+ error in graph	M+ max error
1	0.000%	0.000%	0.000%	0.000%
0.9	0.172%	0.007%	8.279%	0.129%
0.8	0.086%	0.010%	1.442%	0.411%
0.7	-0.162%	-0.001%	-8.481%	0.224%
0.6	-0.579%	-0.063%	-23.860%	-1.005%
0.5	-0.206%	-0.121%	-18.672%	-2.778%
0.4	0.701%	-0.011%	11.164%	-3.744%
0.3	0.854%	0.157%	41.860%	-3.068%
0.2	0.619%	0.351%	94.853%	-2.854%
0.1	-2.664%	0.458%	108.875%	-5.160%

Seventh Order Polynomial Approximation				
c/L	Δ error In graph	Δ max error	M+ error in graph	M+ max error
1	0.000%	0.000%	0.000%	0.000%
0.9	0.109%	0.007%	5.682%	0.129%
0.8	0.059%	0.010%	-1.893%	0.209%
0.7	-0.240%	-0.009%	-14.604%	0.030%
0.6	-0.207%	-0.042%	-6.521%	-0.143%
0.5	0.056%	-0.026%	13.036%	-0.544%
0.4	0.525%	0.066%	34.944%	-2.048%
0.3	0.665%	0.076%	13.035%	-3.706%
0.2	-0.733%	-0.042%	-52.946%	-2.450%
0.1	-2.824%	-0.228%	-132.625%	-1.674%

Eighth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error in graph	M+ max error
1	0.000%	0.000%	0.000%	0.000%
0.9	0.029%	-0.001%	1.633%	-0.032%
0.8	-0.077%	0.002%	-7.956%	0.035%
0.7	-0.017%	0.007%	-1.643%	0.315%
0.6	0.133%	-0.006%	15.180%	0.119%
0.5	0.056%	-0.026%	13.036%	-0.544%
0.4	-0.189%	0.013%	-16.294%	-0.840%
0.3	-0.139%	0.045%	-37.868%	-2.273%
0.2	0.511%	-0.004%	13.571%	-3.342%
0.1	-0.249%	-0.089%	106.610%	-0.567%

Nineth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error in graph	M+ max Error
1	0.000%	0.000%	0.000%	0.000%
0.9	0.013%	-0.001%	0.679%	-0.032%
0.8	-0.062%	0.002%	-6.709%	0.022%
0.7	-0.001%	0.003%	4.628%	0.089%
0.6	0.112%	-0.006%	13.019%	0.161%
0.5	-0.026%	-0.008%	-11.967%	0.054%
0.4	-0.160%	0.015%	-21.158%	-0.856%
0.3	0.090%	0.005%	24.438%	-1.102%
0.2	0.413%	-0.014%	27.563%	-2.976%
0.1	-0.496%	0.014%	-83.283%	-0.589%

TABLE 1 – TAYLOR APPROXIMATION FOR A RECTANGULAR PRESSURE

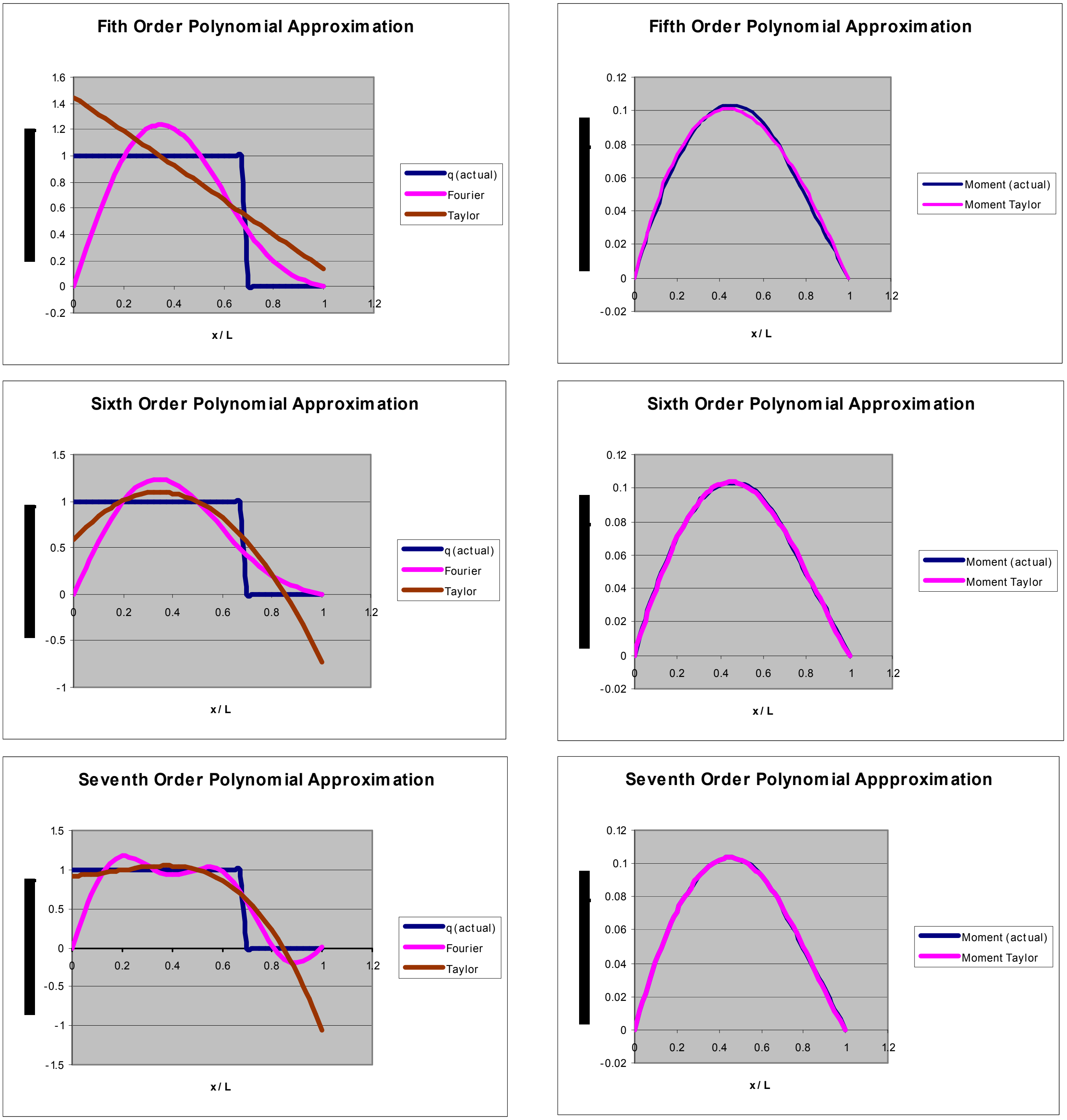

FIGURE 11A – TAYLOR APPROXIMATION FOR A RECTANGULAR PRESSURE c/L = .7

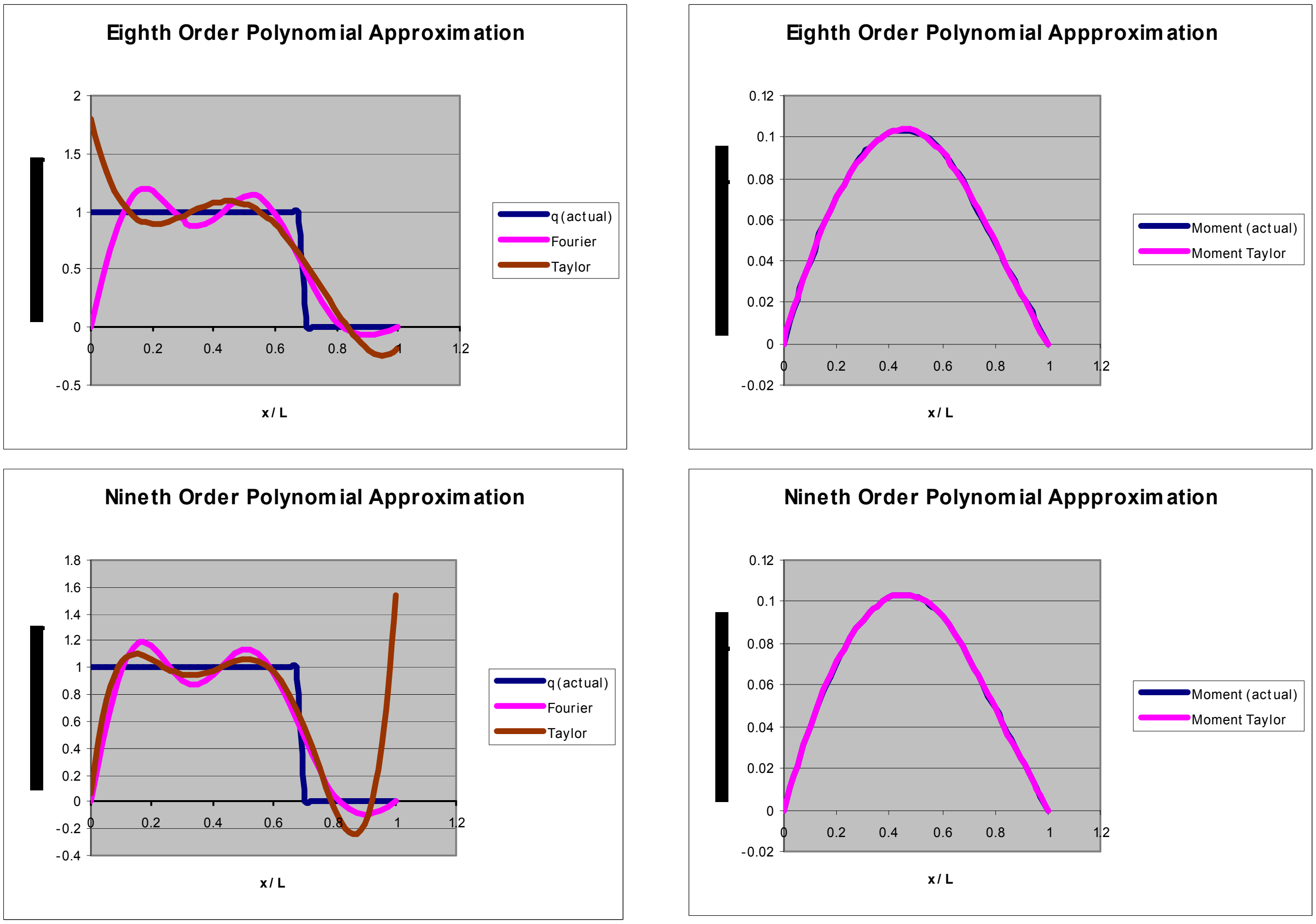

FIGURE 11B – TAYLOR APPROXIMATION FOR A RECTANGULAR PRESSURE c/L = .7

Fifth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error in graph	M+ max error
~1.0	-16.04%	2.90%	39.90%	-18.51%
0.9	-8.34%	2.49%	-125.4%	-14.59%
0.8	-4.05%	1.43%	-90.02%	-12.81%
0.7	1.04%	0.00%	7.15%	-14.98%
0.6	3.82%	-1.25%	49.74%	-19.15%
0.5	2.85%	-1.83%	31.69%	-21.46%
0.4	3.82%	-1.25%	49.74%	-19.15%
0.3	1.04%	0.00%	7.15%	-14.98%
0.2	-4.05%	1.43%	-90.02%	-12.81%
0.1	-8.34%	2.49%	-125.4%	-14.59%

Sixth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error In graph	M+ max error
~1.0	-6.80%	0.52%	99.55%	-9.96%
0.9	-0.17%	0.42%	107.2%	-7.81%
0.8	1.45%	0.19%	37.58%	-11.17%
0.7	1.47%	-0.05%	14.78%	-15.09%
0.6	-1.40%	-0.20%	-69.30%	-13.36%
0.5	-1.06%	-0.32%	-47.81%	-11.53%
0.4	-1.40%	-0.20%	-69.30%	-13.36%
0.3	1.47%	-0.05%	14.78%	-15.09%
0.2	1.45%	0.19%	37.58%	-11.17%
0.1	-0.17%	0.42%	107.2%	-7.81%

Seventh Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error in graph	M+ max error
~1.0	-6.95%	-0.32%	-165.9%	-5.24%
0.9	-1.34%	-0.15%	-93.22%	-5.87%
0.8	1.78%	0.14%	43.16%	-11.51%
0.7	1.22%	0.20%	80.64%	-9.92%
0.6	-1.02%	-0.07%	-34.31%	-9.79%
0.5	-1.06%	-0.32%	-47.81%	-11.53%
0.4	-1.02%	-0.07%	-34.31%	-9.79%
0.3	1.22%	0.20%	80.64%	-9.92%
0.2	1.78%	0.14%	43.16%	-11.51%
0.1	-1.34%	-0.15%	-93.22%	-5.87%

Eighth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error In graph	M+ max error
~1.0	-3.48%	-0.13%	120.96%	-2.20%
0.9	0.38%	-0.05%	54.91%	-6.12%
0.8	0.24%	0.08%	-57.52%	-8.93%
0.7	-0.77%	0.06%	-56.46%	-7.82%
0.6	0.03%	-0.02%	27.46%	-9.31%
0.5	0.62%	-0.11%	61.32%	-7.85%
0.4	0.03%	-0.02%	27.46%	-9.31%
0.3	-0.77%	0.06%	-56.46%	-7.82%
0.2	0.24%	0.08%	-57.52%	-8.93%
0.1	0.38%	-0.05%	54.91%	-6.12%

Nineth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error in graph	M+ max error
~1.0	-3.39%	0.04%	-159.99%	-0.27%
0.9	0.88%	-0.01%	27.14%	-6.67%
0.8	0.57%	-0.02%	83.81%	-6.32%
0.7	-0.82%	0.06%	-59.73%	-7.85%
0.6	-0.17%	0.02%	-27.51%	-6.86%
0.5	0.62%	-0.11%	61.32%	-7.85%
0.4	-0.17%	0.02%	-27.51%	-6.86%
0.3	-0.82%	0.06%	-59.73%	-7.85%
0.2	0.57%	-0.02%	83.81%	-6.32%
0.1	0.88%	-0.01%	27.14%	-6.67%

TABLE 2 – TAYLOR APPROXIMATION FOR POINT LOAD

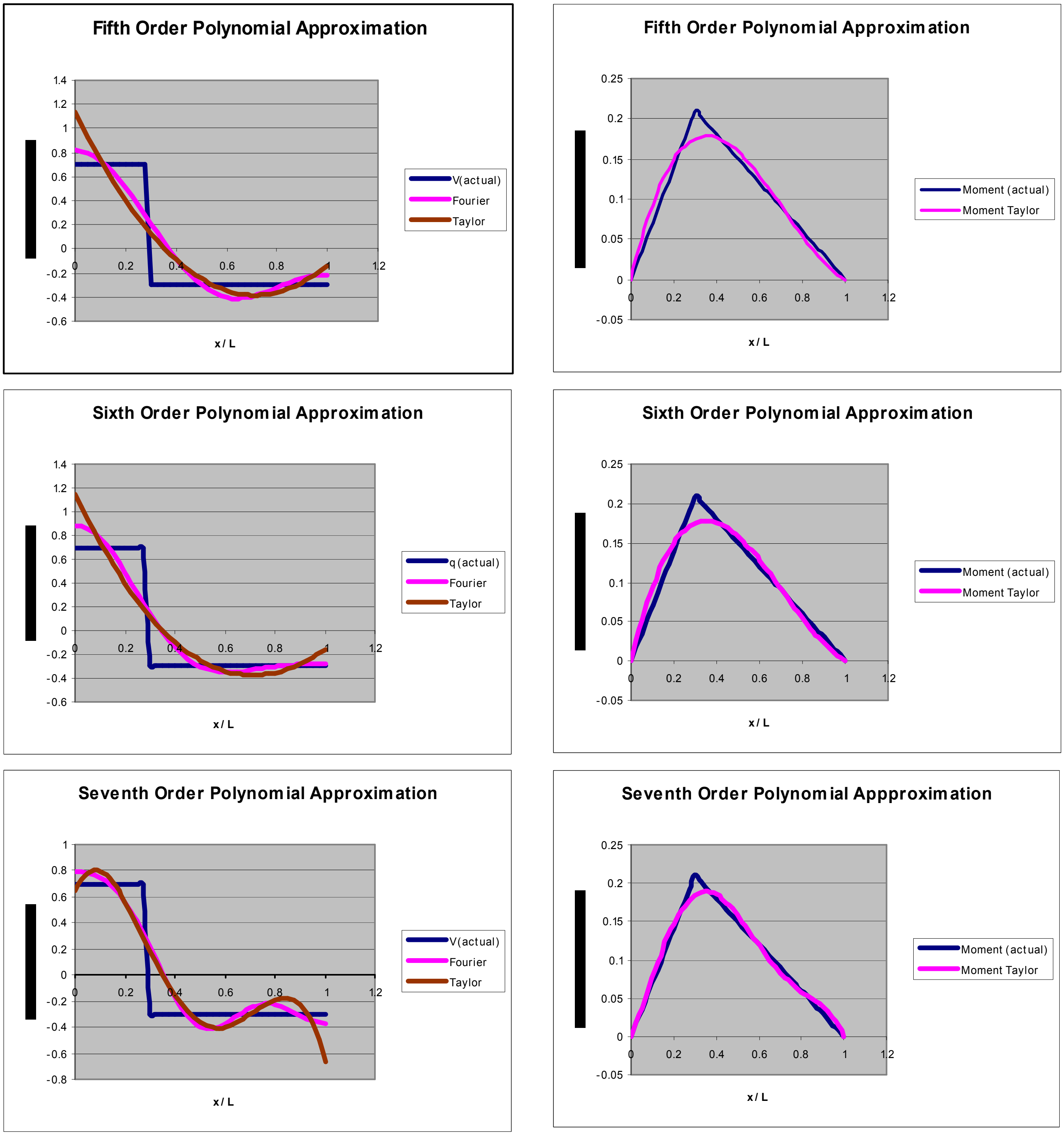

FIGURE 12A – TAYLOR APPROXIMATION FOR A POINT LOAD c/L = 0.3

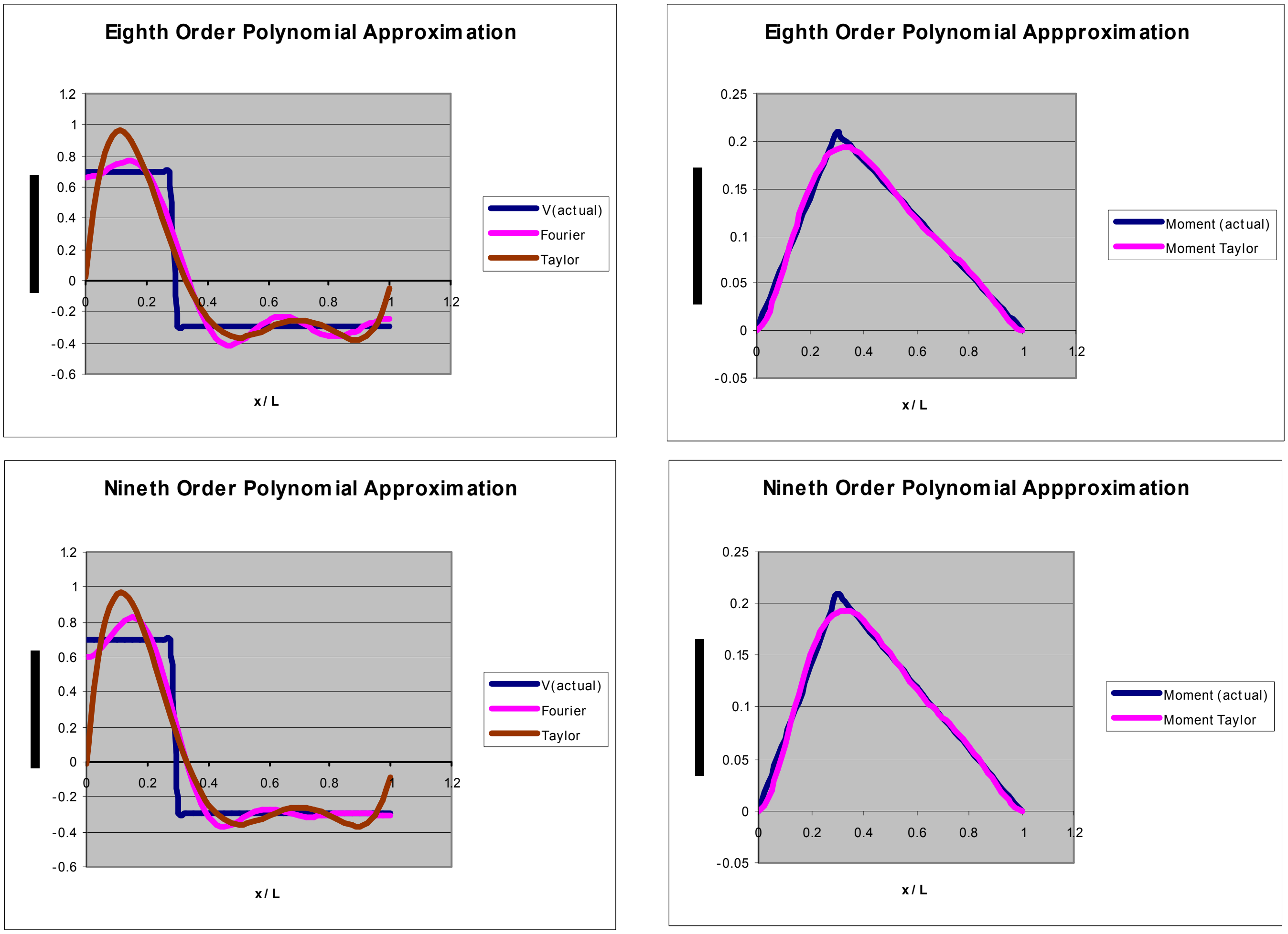

FIGURE 12B – TAYLOR APPROXIMATION FOR A POINT LOAD c/L = 0.3

Fifth Order Polynomial Approximation				
c/L	Δ error In graph	Δ max error	M+ error in graph	M+ max Error
1	0.332%	2.902%	83.868%	-18.512%
0.9	0.151%	1.661%	40.283%	-17.811%
0.8	0.254%	-1.399%	55.860%	-31.887%
0.7	0.309%	-4.884%	47.294%	-40.817%
0.6	0.264%	-4.998%	48.623%	-31.742%
0.5	0.147%	0.399%	43.398%	-33.510%
0.4	0.264%	-4.998%	48.623%	-62.579%
0.3	0.309%	-4.884%	47.294%	-40.817%
0.2	0.254%	-1.399%	55.860%	-31.887%
0.1	0.151%	1.661%	40.283%	-94.090%

Sixth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error in graph	M+ max Error
1	0.151%	0.522%	75.343%	-9.961%
0.9	0.058%	0.281%	51.503%	-15.179%
0.8	0.158%	0.232%	47.402%	-24.721%
0.7	0.110%	0.559%	41.877%	-18.679%
0.6	0.127%	-1.231%	47.297%	-28.493%
0.5	0.147%	0.399%	43.398%	-33.510%
0.4	0.127%	-1.231%	47.297%	-10.201%
0.3	0.110%	0.559%	41.877%	-34.831%
0.2	0.158%	0.232%	47.402%	-24.721%
0.1	0.058%	0.281%	51.503%	-15.179%

Seventh Order Polynomial Approximation				
c/L	Δ error in graph	Δ max Error	M+ error in graph	M+ max Error
1	0.081%	-0.316%	66.387%	-5.242%
0.9	0.069%	0.161%	53.720%	-15.743%
0.8	0.066%	0.736%	39.077%	-13.072%
0.7	0.079%	0.089%	46.368%	-18.872%
0.6	0.086%	0.535%	41.161%	-17.692%
0.5	0.058%	4.890%	39.290%	-13.703%
0.4	0.086%	0.535%	41.161%	-27.868%
0.3	0.079%	0.089%	46.368%	20.425%
0.2	0.066%	0.736%	39.077%	-13.072%
0.1	0.069%	0.161%	53.720%	-15.743%

Eighth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error in graph	M+ max Error
1	0.048%	-0.129%	57.357%	-2.197%
0.9	0.055%	0.089%	50.047%	-12.985%
0.8	0.033%	0.214%	40.231%	-9.766%
0.7	0.064%	0.454%	42.343%	-14.350%
0.6	0.044%	0.793%	39.084%	-10.916%
0.5	0.058%	4.890%	39.290%	-13.703%
0.4	0.044%	0.793%	39.084%	-1.588%
0.3	0.064%	0.454%	42.343%	-1.590%
0.2	0.033%	0.214%	40.231%	-9.766%
0.1	0.055%	0.089%	50.047%	-12.985%

Nineth Order Polynomial Approximation				
c/L	Δ error in graph	Δ max error	M+ error in graph	M+ max error
1	0.030%	0.041%	48.529%	-0.266%
0.9	0.036%	-0.075%	45.368%	-8.437%
0.8	0.040%	0.091%	43.348%	-11.052%
0.7	0.031%	0.447%	36.371%	-6.827%
0.6	0.046%	0.669%	39.901%	-12.012%
0.5	0.031%	2.276%	35.231%	-4.042%
0.4	0.046%	0.669%	39.901%	1.827%
0.3	0.031%	0.447%	36.371%	0.841%
0.2	0.040%	0.091%	43.348%	-11.052%
0.1	0.036%	-0.075%	45.368%	-8.437%

TABLE 3 – TAYLOR APPROXIMATION FOR A MOMENT

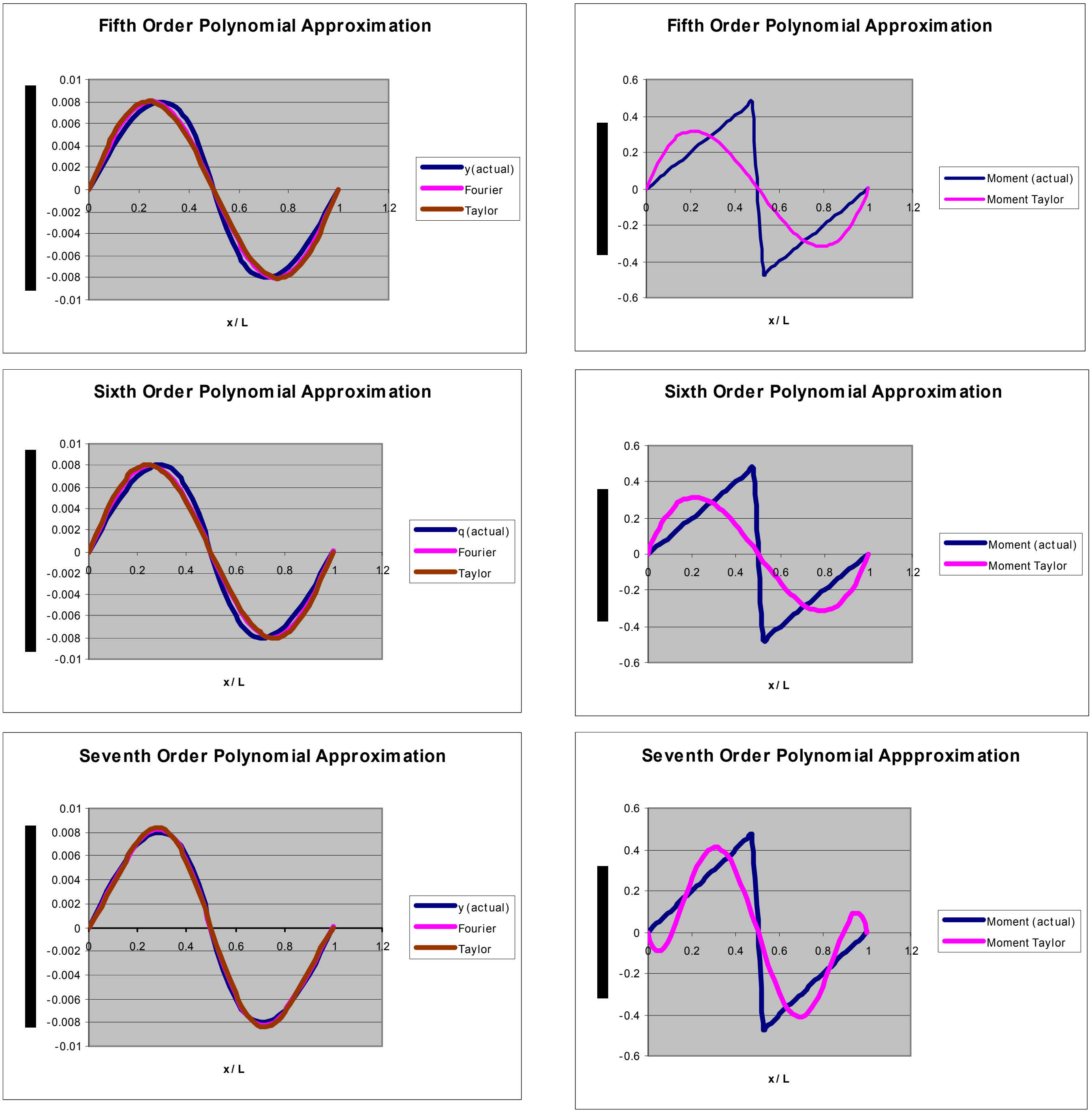

FIGURE 13A – TAYLOR APPROXIMATION FOR A MOMENT c/L = 0.5

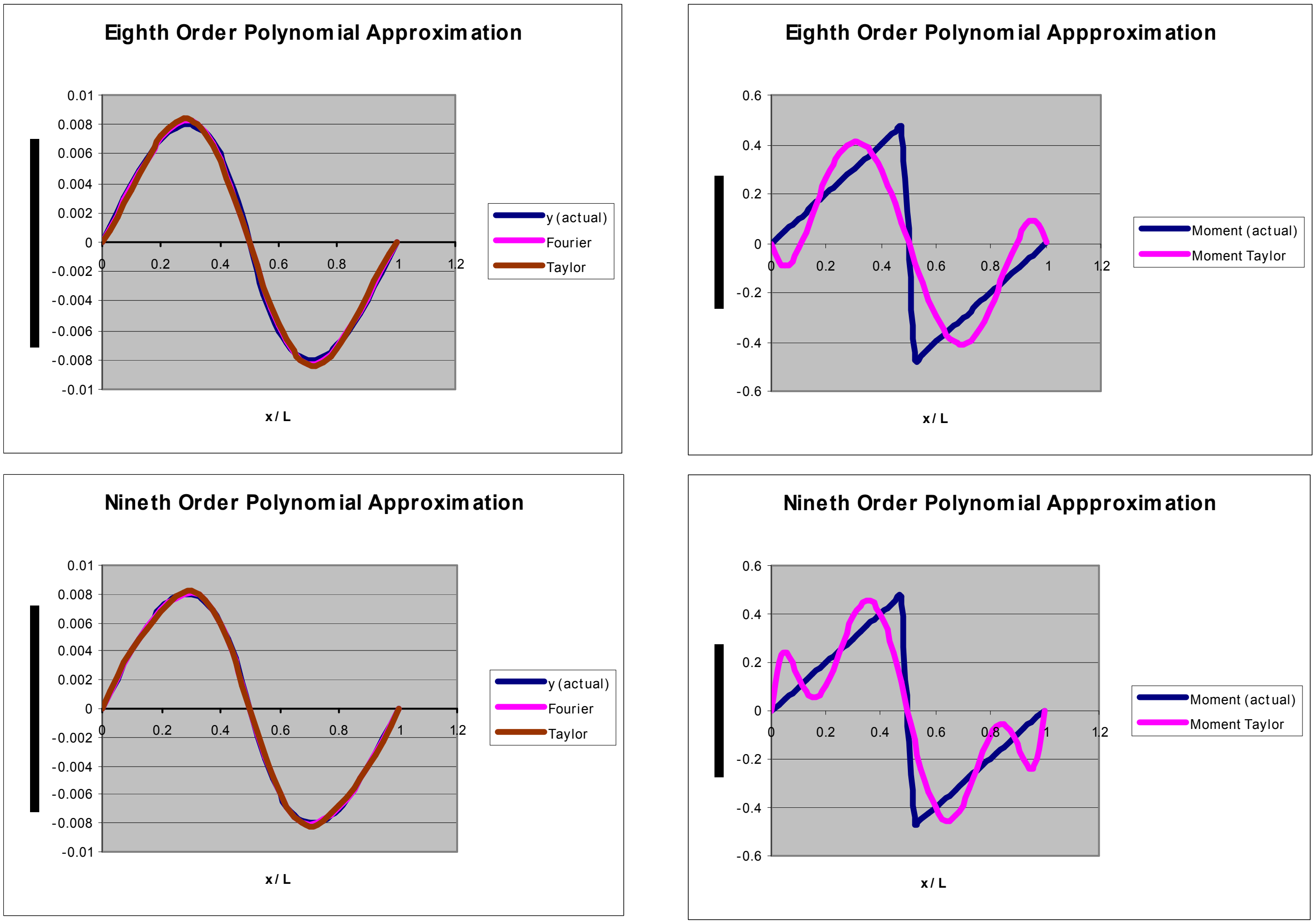

FIGURE 13B – TAYLOR APPROXIMATION FOR A MOMENT c/L = 0.5

From this exercise and examples we know what to expect from our plate solution. By using Timoshenko pp 111 Eq. 133 the point load on a plate can be introduced using the following pressure:

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\frac{m\pi\, x}{a}\sin\frac{n\pi\, y}{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (130)$$

Where

$$a_{mn}=\frac{4P}{ab}\sin\frac{m\pi\,\xi}{a}\sin\frac{n\pi\,\eta}{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (131)$$

Base on the analysis in one dimension we conclude that the shear at the point under the load can be taken as

$$\pm\left[\max\left(\left|Q_x(\xi-\delta_1,\eta+\delta_2)\right|,\left|Q_x(\xi+\delta_1,\eta+\delta_2)\right|,\left|Q_x(\xi-\delta_1,\eta-\delta_2)\right|,\left|Q_x(\xi+\delta_1,\eta-\delta_2)\right|\right)\right]$$

and

$$\pm\max\left(\left|Q_y(\xi+\delta_1,\eta-\delta_2)\right|,\left|Q_y(\xi+\delta_1,\eta+\delta_2)\right|,\left|Q_y(\xi-\delta_1,\eta-\delta_2)\right|,\left|Q_y(\xi-\delta_1,\eta+\delta_2)\right|\right)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (132)$$

for an appropriate increments (δ_1,δ_2) where (δ_1,δ_2) can be found by testing for ultimate values around the point (ξ,η). If we have a load that has been approximated with many point loads then it is best to determine what shear value to use under the load based on design and represent Taylor polynomial approximation as a discontinuous function with point values under the load. If we differentiate the shear to obtain the pressure then the pressure under the load can not be determined using Taylor polynomial as a continuous function and should be taken

$$P=Q_x(\xi-\delta_1,\eta-\delta_2)-Q_x(\xi+\delta_1,\eta+\delta_2) \quad \text{or} \quad P=Q_y(\xi-\delta_1,\eta-\delta_2)-Q_y(\xi+\delta_1,\eta+\delta_2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (133)$$

For the moment pressure equation in the x direction use Eq. 130 and Eq. 131 and let there be two point load in opposite direction at the coordinate $(\xi-\varepsilon,\eta)$ and $(\xi+\varepsilon,\eta)$ then Eq. 131 becomes

$$a_{mn} = \frac{4P}{ab}\sin\frac{n\pi\,\eta}{b}\left[\sin\frac{m\pi\,(\xi-\varepsilon)}{a} - \sin\frac{m\pi\,(\xi+\varepsilon)}{a}\right] = -\frac{8P}{ab}\sin\frac{m\pi\,\varepsilon}{a}\cos\frac{m\pi\,\xi}{a}\sin\frac{n\pi\,\eta}{b}$$

…………………………………… (134)

Now let the moment due to the point load be applied at the point (ξ,η) as $M_{0x} = 2\varepsilon P$. By substituting M_{0x} in Eq. 134 we have:

$$a_{mn} = -\frac{4M_{0x}}{a^2 b}\,m\pi\,\frac{\sin\frac{m\pi\,\varepsilon}{a}}{\frac{m\pi\,\varepsilon}{a}}\cos\frac{m\pi\,\xi}{a}\sin\frac{n\pi\,\eta}{b}$$ ………………………… (135)

Let ε got to zero then we the pressure for a moment in the *x* direction using Eq. 130 with the following:

$$a_{mn} = -\frac{4M_{0x}}{a^2 b}\,m\pi\cos\frac{m\pi\,\xi}{a}\sin\frac{n\pi\,\eta}{b}$$ ……………………………………. (136)

Repeating the above analysis for the pressure for a moment in the *y* direction using Eq. 130 with the following:

$$a_{mn} = -\frac{4M_{0y}}{ab^2}\,n\pi\sin\frac{m\pi\,\xi}{a}\cos\frac{n\pi\,\eta}{b}$$ ……………...………………… (137)

And the four moments surrounding the point under the point of application can be taken as

$$M_x(\xi-\delta_1,\eta+\delta_2), M_x(\xi-\delta_1,\eta-\delta_2), M_x(\xi+\delta_1,\eta+\delta_2), M_x(\xi+\delta_1,\eta-\delta_2)$$

and

$$M_y(\xi+\delta_1,\eta-\delta_2), M_y(\xi-\delta_1,\eta-\delta_2), M_y(\xi+\delta_1,\eta+\delta_2), M_y(\xi-\delta_1,\eta+\delta_2)$$

………………………………….. (138)

for an appropriate increments (δ_1,δ_2) where (δ_1,δ_2) can be found by testing for ultimate values around the point (ξ,η). Thus, it is best to determine the moment values to under the point of application based on design practice and represent Taylor polynomial approximation as a discontinuous function with point values under the load. If we differentiate the moment to obtain the shear then the shear under the load can not be determined using Taylor polynomial as a continuous function and should be taken

$$Q_x = \max\left[\left|-\frac{1}{b}\left[M_{xy}(\xi-\delta_1,\eta-\delta_2) - M_{xy}(\xi-\delta_1,\eta+\delta_2)\right]\right|, \left|-\frac{1}{b}\left[M_{xy}(\xi+\delta_1,\eta-\delta_2) - M_{xy}(\xi+\delta_1,\eta+\delta_2)\right]\right|\right]$$
$$+\max\left[\left|\frac{1}{a}\left[M_x(\xi-\delta_1,\eta+\delta_2) - M_x(\xi+\delta_1,\eta+\delta_2)\right]\right|, \left|\frac{1}{a}\left[M_x(\xi-\delta_1,\eta-\delta_2) - M_x(\xi+\delta_1,\eta-\delta_2)\right]\right|\right]$$

and

$$Q_y = \max\left[\left|-\frac{1}{a}\left[M_{xy}(\xi-\delta_1,\eta+\delta_2) - M_{xy}(\xi+\delta_1,\eta+\delta_2)\right]\right|, \left|-\frac{1}{a}\left[M_{xy}(\xi-\delta_1,\eta-\delta_2) - M_{xy}(\xi+\delta_1,\eta-\delta_2)\right]\right|\right]$$
$$+\max\left[\left|\frac{1}{b}\left[M_y(\xi+\delta_1,\eta-\delta_2) - M_y(\xi+\delta_1,\eta+\delta_2)\right]\right|, \left|\frac{1}{b}\left[M_y(\xi-\delta_1,\eta-\delta_2) - M_y(\xi-\delta_1,\eta+\delta_2)\right]\right|\right]$$

…………………………………… (139)

Comment on selecting δ, δ_1 and δ_2 :

A practical selection of δ, δ_1 and δ_2 are recommended by Professor John Stanton saying the point load in a concrete slab can be seen as a cone propagating in the thickness of the slab. Thus an absolute smallest increment of δ, δ_1 and δ_2 is the thickness of the plate *t* shaped in a circle.

This also becomes a restriction when subdividing a pressure function into point loads for large deflection analysis and it becomes a condition of using the solution in realm of elasticity. Loads that need finer increments then twice the thickness of the slab cannot be approximated into point loads and should be addressed differently when large deflection is the issue. If large deflection is not of concern then Eq. 59 is the best alternative and has been the methods used in standard practice for ages.

Final analysis:
As we can see any load can approximated by point loads for a more conservative solution. However, the increment of divisions has a limiting value as discussed above. The interesting part is the Cartesian solution is it is sufficient solution and other coordinates transformations are not necessary provided the load is contained in the boundary condition. For example for a simply supported circular plate the boundary condition can be satisfied as long as the load is contained in the circle.

Finally for large deflection with a point load P_i , a moment in the x direction M_{ix} and a moment in the y direction M_{iy} at the point of application (ξ_i, η_i) makes the coordinate (ξ_i, η_i) becomes another coordinate (x_i, y_i) in the final large deflection of the plate. Thus, a new set of equations is requires so there is no change in length to the original point (ξ_i, η_i) thus for a rectangular plate we have:

$$\begin{aligned} \xi_i &= \int_0^{x_i} \frac{dx}{\sqrt{1+[w_x(x, y_i)]^2}} \\ \eta_i &= \int_0^{y_i} \frac{dy}{\sqrt{1+[w_y(x_i, y)]^2}} \end{aligned} \qquad \text{(140)}$$

With these additional equations the solution can be found exactly for point loads moments. The solution is similar to finding the coefficients for large deflection of a beams. Thus we start with an initial value for the Taylor polynomial coefficients plus the coefficients (x_i, y_i) and the free boundaries and update numerically, and the solution becomes exact.