Slope Stability Slip Surface Using Variational Methods

By

Farid A. Chouery[1], P.E., S.E.

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Abstract:
In this synopsis, Variational method is used to determine the slope stability slip surface based on ordinary method of slices without pore pressure and not circular for the time being. It is shown that using the ordinary method of slices gives approximately the same shape slip surface for moments equation, so ignoring that the method was derived for circles is adequate. The result shows that geometry and topography of the embankment affects the shape of the slip surface.

Introduction:
The principle of slope stability have been developed over the past seventy years and provide a set of soil mechanics principles from which to approach practical problems. Although the mechanics of slope failure in heap leaching may be difficult to predict, the principles used in a standard of practice examination are relatively straightforward. The proposed method of variation analysis is far better prediction and is a refined method then current methods; the slip surface is prescribed and not guest at. This approach relieves the mathematical uncertainty of what the slip surface is, provided the soil parameters are close to reality. This gave us a better prediction then a circle or log-spiral.

An analysis of slope stability begins with the hypothesis that the stability of a slope is the result of downward or motivating forces (i.e gravitational) and resisting (or upward) forces. These forces act in equal and opposite directions as can be seen in practice. The resisting forces must be greater than the motivating forces in order for a slope to be stable. The relative stability of a slope (or how stable it is at any given time) is typically conveyed by geotechnical engineers through a Factor of Safety F_s defined as follows:

$$F_s = \frac{\sum R}{\sum M}$$

The equation states that the factor of safety is the ratio between the forces/moments resisting (R) movements and the forces/moments motivating (M) movements. When the factor of safety is equal 1.0 a slope has just reached failure conditions. If the factor of safety falls below 1.0 then a failure is imminent, or has already occurred. Factors of safety in the range of 1.3 to 1.5 are considered reasonably safe in many design scenarios. However, the actual factor of safety used in design is influenced by the risk involved as well the certainty with which other variables are known.

[1] Structural, Electrical and Foundation Engineer, FAC Systems Inc. 6738 19th Ave. NW, Seattle, WA 98117

Analysis:
Definitely, if minimizing the safety factor for the ordinary method of slices then it would give a more closer slip surface for a true slip surface. Once the slip surface is prescribed then using a comprehensive framework for limit equilibrium methods of slices developed, for example, by Fredlund te al. (1981) would give a more realistic safety factor. Fredlund methodology can be used for analyses in both circular and non-circular slip surface. Because pore pressure can change the slip surface it will not be considered for the demonstration. So the forgoing analysis is for one condition and it will be checked for a unique situation with φ is zero or for a cohesion material. Still the slip surface derived by Chouery in determining the maximum soil pressure for a smooth wall, currently being published, should be considered since Cullman method is always considered. In this case the slip surface by Chouery assumes a smooth wall witch is appropriate for slope stability of the embankment where there is no friction at the outer surface. Also, the safety factor of the forces and the moments must be considered separately and both have to be minimized and the least one must be considered.

a) <u>Force Analysis</u>
The ordinary method of slices gives the first safety factor of the forces as follows:

$$F = \frac{c\int ds + \gamma \tan\phi \int y\cos\alpha dx}{\gamma\int y\sin\alpha dx} = \frac{\int F_1(x,y,\dot{y})dx}{\int F_2(x,y,\dot{y})dx} = \frac{\int F_1}{\int F_2} \quad \text{.................................... (1)}$$

Where $\dot{y} = -\tan\alpha$

Now the solution must satisfy:

$$F = \frac{Min(\int F_1)}{Max(\int F_2)} \quad \text{.. (2)}$$

Rewrite Eq. 2 as

$$FMax(\int F_2) - Min(\int F_1) = 0 \quad \text{.. (3)}$$

Since $\int F_1$ is a positive number then Eq. 3 can be written as

$$FMax(\int F_2) + Max(-\int F_1) = 0 \quad \text{.. (4)}$$

Or

$$Max(F\int F_2 - \int F_1) = 0 \quad \text{.. (5)}$$

If F is picked then Eq, 5 can be satisfied using Variational method and the constant coefficients of the nonlinear differential equation will satisfy Eq. 5. Now pick a lower number for F then as before and solve Eq. 5 again and continue until Eq. 5 cannot be satisfied anymore and the solution is found for minimum F in Eq. 2. This situation is similar to minimizing Eq. 2 using Lagrange multiplier where seeking to minimize $\int F_1$ with the condition $\int F_2$ is a constant or vise versa. Thus the extreemum that gives the slip surface is

$$Max(\int F_2 + \lambda \int F_1) \quad \text{……………………………………………………………} (6)$$

Where λ is Lagrange multiplier. Thus if $\lambda = 0$ it is Max($\int F_2$) if $\lambda = -\infty$ then it is Min($\int F_1$) and λ is not necessarily $1/F$. λ balance the maximization.

b) Moment Analysis

See Fig 1 for a stationary slip surface a-b. For a moment at point A, where A is to be determined, the equation for the safety factor is:

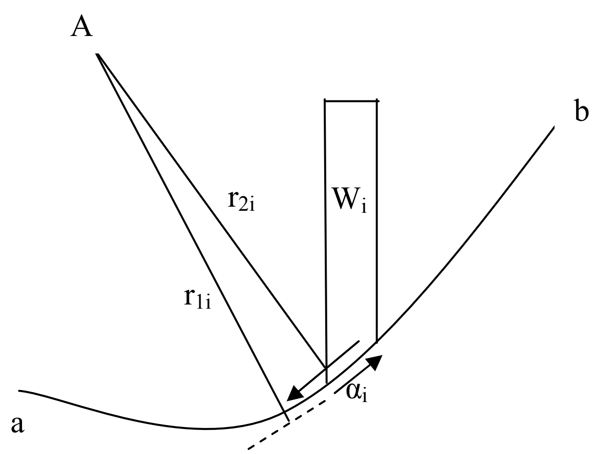

Fig 1 – Slice i taking moments at A

$$F_m = \frac{c\sum r_{1i}\Delta l_i + \tan\phi \sum r_{1i}\cos\alpha_i W_i}{\sum r_{2i}\sin\alpha_i W_i} \quad \text{…………………………………………………}(7)$$

Or

$$F_m = \frac{r_1 \int F_1}{r_2 \int F_2} \quad \text{.. (8)}$$

Where r_1 and r_2 are the resultant radial distances. With many respect $r_{1i} \approx r_{2i}$. Thus

$$F_m = \frac{Min(r_1 \int F_1)}{Max(r_2 \int F_2)} = \frac{Min(r_1) \times Min(\int F_1)}{Max(r_2) \times Max(\int F_2)} \quad \text{.. (9)}$$

Min($\int F_1$) is two parts the $\int ds$ and $\int y\cos\alpha\, dx$. The minimum $\int ds$ is a line thus it forces the slip surface to be less concave upward and r_1 is reduced to a minimum, $\int y\cos\alpha\, dx$ is minimum when W_i (or y) is reduced giving α_i to increase and $\cos\alpha_i$ is to decreased thus again r_1 is reduced to a minimum since the shape is less concave upward. Thus Min($\int F_1$) gives r_1 at minimum anyway.

Max($\int F_2$) is maximum when $\int y\sin\alpha\, dx$ increase or W_i (or y) increase gives a decrease in α_i . Thus r_2 is increase for some constant α_i because the slip surface is becoming more concave upward. We note if we kept W_i (or y) constant and moved α_i up or down by a small increment then $r_{2\mathrm{i}}$ increase for some slices and decrease for others and r_2 remain relatively the same. Thus the bulk of the maximization of r_2 is increasing W_i and not increasing $\sin\alpha_i$. Also, if increase concavity after Max($\int F_2$) is reached then $\int y\sin\alpha\, dx$ will decrease and r_2 will increase but remain relatively the same and the product $r_2\int F_2$ may decrease since $\int F_2$ decrease. Thus Max($\int F_2$) gives r_2 at the approximately maximum anyway. In reality Eq. 9 will give a different slip surface then Eq. 2.

This assumption of concavity upwardly is based on physics and experiments since if the slip surface is concave downward for a given c and φ is not possible. Thus Eq. 9 becomes:

$$F_m \approx \frac{r_1 \times Min(\int F_1)}{r_2 \times Max(\int F_2)} \quad \text{.. (10)}$$

Which reduce to maximizing Eq. 6. Thus the slip surface for Eq. 2 and Eq. 10 has the same shape only the constant coefficients from solving the non linear differential equation of Eq. 6 are different and that makes sense physically. Thus, the only slip surfaces to be investigated is from Eq. 6.

Eq. 6 can be written as

$$-\gamma \int_{x_1}^{x_2} \frac{\dot{y}[y + f(x)]}{\sqrt{1+\dot{y}^2}} dx + \lambda \left[c\int_{x_1}^{x_2} \sqrt{1+\dot{y}^2}\, dx + \gamma \tan\phi \int_{x_1}^{x_2} \frac{y + f(x)}{\sqrt{1+\dot{y}^2}} dx \right] \quad \text{........................... (11)}$$

Where $f(x)$ is the function representing the topography. We apply the Euler equation

$$\frac{\partial \Re}{\partial y}-\frac{d}{dx}\left[\frac{\partial \Re}{\partial \dot{y}}\right]=0 \quad \text{and} \quad \Re=-\gamma\frac{\dot{y}(y+f(x))}{\sqrt{1+\dot{y}^2}}+\lambda\left[c\sqrt{1+\dot{y}^2}+\gamma\tan\phi\frac{y+f(x)}{\sqrt{1+\dot{y}^2}}\right] \quad \ldots\ldots\ldots\ldots (12)$$

As before with Chouery's article we start with a Fourier series representation of $f(x)$ and convert to a Taylor series by inverting the matrix representation and solve Eq. 12 with a polynomial $y=\sum_{n=0}^{m}a_n x^n$ where m gives a good approximation of the curve where the coefficient a_{m+1} diminish if substituting x by x/y_0 where y_0 is the maximum height of the embankment.

We conclude that $f(x)$ affects the slip surface or the geometry and topography affects it but do not know how much. Also, the slip surface is prescribed not guessed at. Examples will follow.

<u>Examples 1:</u>

The solution for $f(x) = 0$ and $\varphi = 0$ will be sought. Note: the ordinary method of slices gives the same answer as other methods. Fig 2 shows the embankment for $\lambda = 0$.

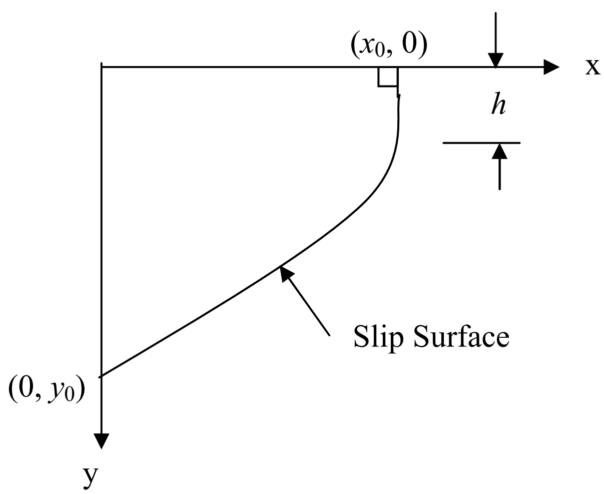

Fig 2 Slip Surface for $f(x) = 0$, $\varphi = 0$ and $\lambda = 0$

From Eq. 12 yields,

$$\Re=-\gamma\frac{\dot{y}y}{\sqrt{1+\dot{y}^2}}+\lambda\left[c\sqrt{1+\dot{y}^2}\right] \quad \text{and} \quad \Re-\dot{y}\frac{\partial \Re}{\partial \dot{y}}=h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

If set $\lambda = 0$ the solution yields:

$$\dot{y} = \pm\frac{h^{\frac{1}{3}}}{\sqrt{y^{\frac{2}{3}} - h^{\frac{2}{3}}}} \quad \text{and} \quad \dot{x} = \pm h^{-\frac{1}{3}}\sqrt{y^{\frac{2}{3}} - h^{\frac{2}{3}}} \quad \ldots\ldots (14)$$

Where h is a constant and the slip surface becomes:

$$x = \pm h^{-\frac{1}{3}}\int\sqrt{y^{\frac{2}{3}} - h^{\frac{2}{3}}}\,dy + k \quad \ldots\ldots (15)$$

Where k is a constant of integrations. The factor of safety is:

$F = \dfrac{3.845c}{\gamma y_0}$ and $h = 0.1733\, y_0$, x_0 (at $y = h$) is $x_0 = 0.866\, y_0$, $\dot{y}\big|_{y=y_0} = -0.6716$ or 33.885°

Where the vertical portion at h in Fig. 2 is taken as having a cohesion c. Note a circle gives 3.83 instead of 3.845 thus $\lambda \neq 0$ and will be calculated in the next examples. Also: note the slip surface does not look like a circle or log spiral and is 90 degrees to the top surface. F_m exact is not calculated in this exercise.

Example 2:

When considering $\lambda \neq 0$ in example 1 and using Euler equation of Eq. 12 it yields:

$$-\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} - \frac{d}{dx}\left[-\frac{y}{\sqrt{1+\dot{y}^2}} + \frac{y\dot{y}^2}{\left(1+\dot{y}^2\right)^{\frac{2}{3}}} + \lambda\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right] = 0 \quad \ldots\ldots (16)$$

Where in here λ is a new constant. Eq. 16 yields:

$$\ddot{y} = -\frac{\dot{y}^3 + \dot{y}^5}{3y\dot{y} - \lambda(1+\dot{y}^2)} \quad \ldots\ldots (17)$$

Working with a forth order Taylor polynomial approximation series so let

$$\frac{y}{y_0} = 1 + a_1\left(\frac{x}{y_0}\right) + a_2\left(\frac{x}{y_0}\right)^2 + a_3\left(\frac{x}{y_0}\right)^3 + a_4\left(\frac{x}{y_0}\right)^4 \quad \ldots\ldots (18)$$

$$\frac{\dot{y}}{y_0} = \frac{a_1}{y_0} + \frac{2a_2}{y_0}\left(\frac{x}{y_0}\right) + \frac{3a_3}{y_0}\left(\frac{x}{y_0}\right)^2 + \frac{4a_4}{y_0}\left(\frac{x}{y_0}\right)^3$$

or ……………………………………….. (19)

$$\dot{y} = a_1 + 2a_2\left(\frac{x}{y_0}\right) + 3a_3\left(\frac{x}{y_0}\right)^2 + 4a_4\left(\frac{x}{y_0}\right)^3$$

$$\ddot{y} = \frac{2a_2}{y_0} + \frac{6a_3}{y_0}\left(\frac{x}{y_0}\right) + \frac{12a_4}{y_0}\left(\frac{x}{y_0}\right)^2 \quad \text{………………………………………………… (20)}$$

$$\dddot{y} = \frac{6a_3}{y_0^2} + \frac{24a_4}{y_0^2}\left(\frac{x}{y_0}\right) \quad \text{……………………………………………………………… (21)}$$

Now let

$$g\left(\frac{x}{y_0}\right) = -\frac{\dot{y}^3 + \dot{y}^5}{3y\dot{y} - \lambda(1 + \dot{y}^2)} \quad \text{………………………………………………………… (22)}$$

$g(x/y_0)$ can be expressed with a Taylor approximation as

$$g\left(\frac{x}{y_0}\right) = g(0) + \frac{1}{1!}g'(0)\left(\frac{x}{y_0}\right) + \frac{1}{2!}g''(0)\left(\frac{x}{y_0}\right)^2 \quad \text{…………………..……………………… (23)}$$

From Eq. 17, 20 and 22

$$\frac{2a_2}{y_0} = g(0) = -\frac{a_1^3 + a_1^5}{3y_0a_1 - \lambda(1 + a_1^2)}$$

or …………………………………………………….... (24)

$$a_2 = \frac{y_0}{2}\left[-\frac{a_1^3 + a_1^5}{3y_0a_1 - \lambda(1 + a_1^2)}\right]$$

$$\frac{6a_3}{y_0} = \frac{1}{1!}g'(0) = \frac{3\dot{y}^5 + 3\dot{y}^7 - 6y(\dot{y}^3 + 2\dot{y}^5)\ddot{y} + 3\lambda(\dot{y}^2 + 2\dot{y}^4 + \dot{y}^6)\ddot{y}}{\left[3y\dot{y} - \lambda(1+\dot{y}^2)\right]^2}$$

or

$$a_3 = \frac{y_0}{6}\left\{\frac{3a_1^5 + 3a_1^7 - 12(a_1^3 + 2a_1^5)a_2 + 6\lambda(a_1^2 + 2a_1^4 + a_1^6)\dfrac{a_2}{y_0}}{\left[3y_0a_1 - \lambda(1+a_1^2)\right]^2}\right\}$$

……………………………………………………… (25)

$$\frac{12a_4}{y_0} = \frac{1}{2!}g''(0) = \frac{\begin{array}{c}15\dot{y}^4\ddot{y} + 21\dot{y}^6\ddot{y} - 6\dot{y}(\dot{y}^3 + 2\dot{y}^5)\ddot{y} - 6y(3\dot{y}^2 + 10\dot{y}^4)\ddot{y}^2 - 6y(\dot{y}^3 + 2\dot{y}^5)\dddot{y}\\ + 3\lambda(2\dot{y} + 8\dot{y}^3 + 6\dot{y}^5)\ddot{y}^2 + 3\lambda(\dot{y}^2 + 2\dot{y}^4 + \dot{y}^6)\dddot{y}\end{array}}{\left[3y\dot{y} - \lambda(1+\dot{y}^2)\right]^2}$$
$$- \frac{2\left[3\dot{y}^5 + 3\dot{y}^7 - 6y(\dot{y}^3 + 2\dot{y}^5)\ddot{y} + 3\lambda(\dot{y}^2 + 2\dot{y}^4 + \dot{y}^6)\ddot{y}\right]\left[3\dot{y}^2 + 3y\ddot{y} - 2\lambda\dot{y}\ddot{y}\right]}{\left[3y\dot{y} - \lambda(1+\dot{y}^2)\right]^3}$$

or

$$a_4 = \frac{y_0}{24}\left\{\frac{\begin{array}{l}(30a_1^4 + 42a_1^6)\dfrac{a_2}{y_0} - 12a_1(a_1^3 + a_1^5)\dfrac{a_2}{y_0} - 24y_0(3a_1^2 + 5a_1^4)\left(\dfrac{a_2}{y_0}\right)^2 - 36(a_1^3 + a_1^5)\dfrac{a_3}{y_0}\\ + 12\lambda(2a_1 + 8a_1^3 + 6a_1^5)\left(\dfrac{a_2}{y_0}\right)^2 + 18\lambda(a_1^2 + 2a_1^4 + a_1^6)\dfrac{a_3}{y_0^2}\end{array}}{\left[3y_0a_1 - \lambda(1+a_1^2)\right]^2}\right.$$
$$\left. - \frac{2\left[3a_1^5 + 3a_1^7 - 12(a_1^3 + 2a_1^5)a_2 + 6\lambda(a_1^2 + 2a_1^4 + a_1^6)\dfrac{a_2}{y_0}\right]\left[3a_1^2 + 6a_2 - 4\lambda a_1\dfrac{a_2}{y_0}\right]}{\left[3y_0a_1 - \lambda(1+a_1^2)\right]^3}\right\}$$

……………………………………………………………….(26)

Thus pick a_1 and λ given y_0 find a_2 from Eq. 24 substitute in Eq. 25 and find a_3 and substitute in Eq. 26 and find a_4. Set $y = 0$ in Eq. 18 and find the proper root x_0 and minimize F in Eq. 1 using Eq. 18 and 19.

$$F = \frac{c\int_0^{x_0} \sqrt{1+\dot{y}^2}\,dx}{-\gamma\int_0^{x_0} \frac{\dot{y}y}{\sqrt{1+\dot{y}^2}}\,dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (27)$$

To compare with example 1, the solution for a forth degree polynomial for $\lambda = 0$ is:

$y_0 = 20$

$a_1 = -0.85$ (on bottom 40.36°) in example 1 it was -0.6716 (33.885°)

$a_2 = -0.207$

$a_3 = -7.728 \times 10^{-3}$

$a_4 = -3.469 \times 10^{-4}$

x_0 (at $y = 0$) = 0.949 y_0 in example 1 it was 0.866 y_0 (at surface tanα = 1.289 or 52.19°)

$$F = \frac{3.898c}{\gamma y_0}$$

So the result shows the forth order polynomial approximation is 1.5% higher than example 1. In order to be closer to F of example 1 more terms are needed in the Taylor series polynomial. All numbers are close and not obtained from an algorithm so there could be computation errors or an error in the derivation. The effect of λ has a solution as follows:

$y_0 = 20$

$\lambda = -19.55$

$a_1 = -0.683$ (on bottom 34.33°)

$a_2 = -0.38$

$a_3 = -0.027$

$a_4 = -8.035 \times 10^{-4}$

x_0 (at $y = 0$) = 0.94 y_0 (at surface tanα = 1.526 or 56.76°)

$$F = \frac{3.855c}{\gamma y_0}$$

Or F is lower for using λ but not quite as example 1 because more terms are needed in the polynomial.

Example 3:

We seek a different solution than Example 2.Thus considering $\lambda \neq 0$ in example 1 and using Euler equation of Eq. 12 it yields:

$$-\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}-\frac{d}{dx}\left[-\frac{y}{\sqrt{1+\dot{y}^2}}+\frac{y\dot{y}^2}{\left(1+\dot{y}^2\right)^{\frac{2}{3}}}+\lambda\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right]=0 \quad \text{.................................} \ (28)$$

Where in here λ is a new constant. Eq. 28 yields:

$$\ddot{x}=\frac{\dot{x}^2+1}{3y\dot{x}-\lambda(\dot{x}^2+1)} \quad \text{...} \ (29)$$

Working with a forth order Taylor polynomial approximation series so let

$$\frac{x}{y_0}=1+a_1\left(\frac{y}{y_0}\right)+a_2\left(\frac{y}{y_0}\right)^2+a_3\left(\frac{y}{y_0}\right)^3+a_4\left(\frac{y}{y_0}\right)^4 \quad \text{..} \ (30)$$

$$\frac{\dot{x}}{y_0}=\frac{a_1}{y_0}+\frac{2a_2}{y_0}\left(\frac{y}{y_0}\right)+\frac{3a_3}{y_0}\left(\frac{y}{y_0}\right)^2+\frac{4a_4}{y_0}\left(\frac{y}{y_0}\right)^3$$

or .... (31)

$$\dot{x}=a_1+2a_2\left(\frac{y}{y_0}\right)+3a_3\left(\frac{y}{y_0}\right)^2+4a_4\left(\frac{y}{y_0}\right)^3$$

$$\ddot{x}=\frac{2a_2}{y_0}+\frac{6a_3}{y_0}\left(\frac{y}{y_0}\right)+\frac{12a_4}{y_0}\left(\frac{y}{y_0}\right)^2 \quad \text{..} \ (32)$$

$$\dddot{y}=\frac{6a_3}{y_0^2}+\frac{24a_4}{y_0^2}\left(\frac{y}{y_0}\right) \quad \text{..} \ (33)$$

Now let

$$g\left(\frac{y}{y_0}\right)=-\frac{\dot{y}^3+\dot{y}^5}{3y\dot{y}-\lambda(1+\dot{y}^2)} \quad \text{...} \ (22)$$

$g(y/y_0)$ can be expressed with a Taylor approximation as

$$g\left(\frac{y}{y_0}\right) = g(0) + \frac{1}{1!}g'(0)\left(\frac{y}{y_0}\right) + \frac{1}{2!}g''(0)\left(\frac{y}{y_0}\right)^2 \quad \text{...} (35)$$

From Eq. 29, 32 and 34

$$\frac{2a_2}{y_0} = g(0) = -\frac{1}{\lambda}$$

or .... (36)

$$a_2 = -\frac{y_0}{2\lambda}$$

$$\frac{6a_3}{y_0} = \frac{1}{1!}g'(0) = \frac{9y\dot{x}^2\ddot{x} - 3\dot{x}(1+\dot{x}^2) + 3y\dddot{x}}{\left[3y\dot{x} - \lambda(1+\dot{x}^2)\right]^2}$$

or

$$a_3 = -\frac{y_0}{2}\left[\frac{a_1}{\lambda^2(1+a_1^2)}\right]$$

.... (37)

$$\frac{12a_4}{y_0} = \frac{1}{2!}g''(0) = \frac{18y\dot{x}\ddot{x}^2 + 3y\ddddot{x}}{\left[3y\dot{x} - \lambda(1+\dot{x}^2)\right]^2} - \frac{2\left[9y\dot{x}^2\ddot{x} - 3\dot{x}(1+\dot{x}^2) + 3y\dddot{x}\right]\left[3\dot{x} - 3y\ddot{x} - 2\lambda\dot{x}\ddot{x}\right]}{\left[3y\dot{x} - \lambda(1+\dot{x}^2)\right]^3}$$

or

$$a_4 = -\frac{5y_0}{4\lambda^3}\frac{a_1}{(1+a_1^2)^2}$$

....(38)

Thus pick a_1 and λ given y_0 find a_2 from Eq. 36, find a_3 in Eq. 37 and find a_4 in Eq. 38. Set $\dot{x} = 0$ in Eq. 31 and find the proper root y and make sure it is negative and minimize F in Eq. 1 using Eq. 31.

$$F = \frac{c\int_0^{y_0} \sqrt{1+\dot{x}^2}\, dy}{-\gamma\int_0^{y_0} \frac{\dot{x}y}{\sqrt{1+\dot{x}^2}}\, dy} \quad \text{......} \quad (39)$$

The effect of λ has a solution as follows:

$y_0 = 20$

$\lambda = 17.5$

$a_1 = -0.35$ (at surface $\tan\alpha = 2.857$ or 70.71° this is closer to 90° similar to example 1)

$a_2 = -0.571$

$a_3 = 0.01$

$a_4 = -4.535 \times 10^{-4}$

$a_0 = - a_1 - a_2 - a_3 - a_4 = 0.912$ or $x_0 = 0.912\, y_0$ (it was $0.938\, y_0$ in example 2)

$\dot{x}\big|_{y=y_0} = a_1 + 2a_2 + 3a_3 + 4a_4 = -1.464$ or $\alpha_0 = 34.335^{\circ}$ (at bottom $\alpha_0 = 34.33^{\circ}$ in Example 2)

$$F = \frac{3.808c}{\gamma y_0}$$

Thus this polynomial approximation gave a factor for F of 3.808 less than a circle of a F factor of 3.83 and the Variational method works.

Note: λ takes a positive value due to:

$$\frac{1}{-\lambda}\int y \sin\alpha dx = \frac{1}{-\lambda}\int \frac{y}{\sqrt{1+\dot{x}^2}} dx = \frac{1}{\lambda}\int \frac{y\dot{x}}{\sqrt{1+\dot{x}^2}} dy = -\frac{1}{-\lambda}\int \frac{y\dot{y}}{\sqrt{1+\dot{y}^2}} dx$$

Conclusion:

Geometry affects the shape of the slip surface but we do not know by how much. Also, from the example shown one may conclude that maximizing Max($\int F_2$), which is independent from c and φ, is all that is needed and just add a safety factor to it. This is not correct the example shows otherwise. Also, this is for one condition, generalization per one condition is not acceptable and when including φ a difference from a circle and a log spiral can be 13%. For example the slip surface can be circular on the bottom and linear at the top so what is the correct slip surface? Thus, deriving the correct slip surface is more desirable since the shape affect the safety factor. Note: in finite element, using shearing strength reduction, relies on Poisson's ratio and have not produce significant difference over the limit equilibrium method and the slip surface runs through the elements and is not well defined and no plasticity. The slip surface is somewhere in the elements relying on the size of the element. Finite element is still an approximation where Variational Method represents the exact slip surface given the Taylor series higher terms diminish. It is expected with Variational method there will be a difference and produce a more reliable safety factors. If we say the hill is safe we better make sure.