Kp for a Wall with Friction with Exact Slip Surface

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Introduction:
We know from before (http://www.facsystems.com/Slip.pdf) the best curve for Kp with no friction on the wall is a passive line proved by variations. Shields and Tolunay (1973)[2] did a modified Terzaghi (1943)[20] analysis for passive pressure in granular material on a wall with positive friction. In their solution they used the method of slices and assumed the shear between slices to be zero on each slice except for the first slice next to the wall. The slip surface they used is a logarithmic spiral and their results were close to experimental findings. Basudhar and Madhav (1980)[1] repeated the analysis and included cohesion and pore pressure. In their analysis they did not use the same initial angle α_w at the wall between the logarithmic spiral and the horizontal; they obtained α_w from minimizations of Kp. It will be shown the Kp can be obtain with the exact slip surface from variation methods and the initial angle α_w at the wall between the new slip surface and the horizontal cannot be obtained from minimizations. Angle α_w from Shields and Tolunay (1973)[2] is correct in Zone II in Fig. 1. The proposed α_w at the wall is used from modifications of Shields and Tolunay in Zone I in Fig. 1 and matching experiment. So

$$\alpha_w = 0.5 * \left\{ \cos^{-1}\left[\cos(\phi - \delta) - \frac{\sin(\phi - \delta)}{\tan\phi} \right] - \phi - \delta \right\} \quad \text{for } \alpha_w \geq 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

$$\alpha_w = 0.5 * \left\{ \cos^{-1}\left[\cos(\phi - \delta) - \frac{\sin(\phi - \delta)}{\tan\phi} \right] - \phi - \delta \right\} \left[\frac{\frac{\pi}{4} + \frac{\phi}{2}}{\phi} \right] \quad \text{for } \alpha_w < 0 \ \ldots\ldots\ldots\ldots\ldots \ (2)$$

Where, ϕ is the internal friction angle of the soil and δ is the positive wall friction with the soil. Eq. 2 was modified from Eq. 1 to match Terzaghi (1943)[20] bearing capacity assumption (see Fig. 1 Zone I). (This choice was chosen out respect for Terzaghi; in reality Shields and Tolunay angle is ok the way it is and it may not make much difference with the adjustment). The proposed slip surface is obtained by implementing Shields and Tolunay (1973)[2] assumption that the friction is dissipated all in the first slice using the method of slices. The resulting solution will be shown to be an absolute lower bound to Kp, lower than experimental findings indicating the wall friction dissipates to more then one slice at the wall. However, if someone would like to research the friction dissipating in more than one slice, our derived slip surface still must be used in the zone with no friction between slices (If needed preliminary equation has been prepared and can be provided upon request). Thus, the following derived slip surface is exact and

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important. To mach experiments we select a specific geometry as an approximation and assume geometrical harmony.

Analysis:

We seek to find a slip surface by variation with the angle α_w at the wall prescribed. Following Terzaghi (1943)[20] we break the slip surface to three zones see Fig. 1. Zone III has a line for the curve with $\alpha_0 = \pi/4 - \phi/2$ as is expected from the variation analysis (http://www.facsystems.com/Slip.pdf). Zone I & II the wedge forces can be expressed as:

$$\int dE = \tan(\psi \pm \phi)\int dW \quad \ldots\ldots (3)$$

Where the total wedge weight of each slice in Zone I or II = $\int dW$ and the total passive force from each slice Zone I or II = $\int dE$. $(\psi \pm \phi)$ is the direction of the resultant force on the bottom of the wedge.

If we minimize the total passive force $\int dE$ with Eq. 3 as a condition, where $(\psi \pm \phi)$ is assumed constant and at the actual value of the effective Kp. So the direction of the resultant at the bottom of the total wedge is constant and is independent from the slip surface variation. This condition comes intuitively. So, we will obtain a slip surface that can have a prescribed α_w.Thus, from (http://www.facsystems.com/Slip.pdf) we have:

$$\int dE = -\gamma\int_0^{x_0} \frac{y' - \tan\phi}{1 + y'\tan\phi} y dx \quad \ldots\ldots (4)$$

$$\int dW = \gamma\int_0^{x_0} y dx \quad \ldots\ldots (5)$$

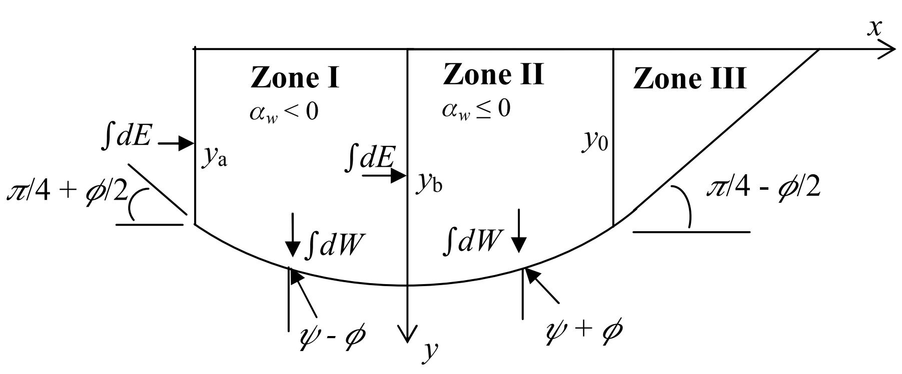

Fig. 1 Slip Surface Zones

Therefore Euler's equation [6] applies where

$$\Re = -\gamma \frac{y' - \tan\phi}{1 + y'\tan\phi} y + \lambda_0 \left[-\gamma \frac{y' - \tan\phi}{1 + y'\tan\phi} y - \gamma y \right] \quad \text{and} \quad \frac{\partial \Re}{\partial y} - \frac{d}{dx}\left(\frac{\partial \Re}{\partial y'} \right) = 0 \qquad (6)$$

which can be written as $\Re - y' \frac{\partial \Re}{\partial y'} = h_0$ .... (7)

and λ_0 is Lagrange multiplier.

Where $\Re$ does not involve x explicitly, and h_0 is a constant.
Substituting in Eq. 7 yields

$$\gamma y \left[-\frac{y' - \tan\phi}{1 + y'\tan\phi} - \frac{\lambda_0}{1 + \lambda_0} - y' \frac{1 + \tan^2\phi}{(1 + y'\tan\phi)^2} \right] = \frac{h_0}{1 + \lambda_0} \qquad (8)$$

Or:

$$\frac{y' - \tan\phi}{y' + \cot\phi} + \lambda - y' \frac{\cot\phi + \tan\phi}{(y' + \cot\phi)^2} = \frac{\bar{c}}{y} \qquad (9)$$

Where, $\lambda = \frac{\lambda_0}{1 + \lambda_0} \tan\phi$ and $\bar{c} = \frac{h_0}{\gamma} \frac{\tan\phi}{1 + \lambda_0}$.

If $h_0 = 0$, $\lambda_0 = 0$ then when solving for y' in Eq. 9, we have the classic passive line with $\alpha_w = \pi/4 - \phi/2$ and Kp = $\tan^2(\pi/4 + \phi/2)$. Thus, the line slip surface becomes a special condition of Eq. 9.

Substitute $x' = dx/dy = 1/y'$ in Eq. 9 and solve for x' yields:

$$x' = -\tan\phi \left[1 \pm \sqrt{\frac{\tan\phi + \cot\phi}{-\tan\phi + \lambda\cot\phi}} \sqrt{\frac{y}{c - y}} \right] \qquad (10)$$

Where, $c = \frac{\bar{c}\tan\phi}{-\tan\phi + \lambda\cot\phi}$. Now at $y = y_b$, $y' = 0$ and $x' \to \infty$ thus $c = y_b$. Also in Zone II at

$y = y_0$ $x' = -\tan\phi\left[1 + \frac{1}{\sin\phi}\right]$ to mach the line of Zone III. Eq. 10 can be written as:

$$x' = -\tan\phi \left[1 + \frac{1}{\sin\phi} \sqrt{\frac{y_b - y_0}{y_0} \frac{y}{y_b - y}} \right] \qquad (11)$$

Also in Zone I at $y = y_a$ $x' = -\tan\phi\left[1-\frac{1}{\sin\phi}\right]$ to mach the line of Zone IV (not shown , where Terzaghi (1943)[20] explains that zone is for the slip surface for bearing capacity for ideal soil smooth base.) .Eq. 10 can be written as:

$$x' = -\tan\phi\left[1-\frac{1}{\sin\phi}\sqrt{\frac{y_b - y_a}{y_a}\frac{y}{y_b - y}}\right] \quad \text{.......... (12)}$$

Integrating Eq. 11 gives the slip surface for Zone II:

$$x = -\tan\phi(y - y_b) + \frac{1}{\cos\phi}\sqrt{\frac{y_b - y_0}{y_0}}\left\{\sqrt{y_b y - y^2} - 0.5y_b\left[\sin^{-1}\left(\frac{2y}{y_b}-1\right)-\frac{\pi}{2}\right]\right\} \quad \text{.......... (13)}$$

Integrating Eq. 12 gives the slip surface for Zone I:

$$x = -\tan\phi(y - y_b) - \frac{1}{\cos\phi}\sqrt{\frac{y_b - y_a}{y_a}}\left\{\sqrt{y_b y - y^2} - 0.5y_b\left[\sin^{-1}\left(\frac{2y}{y_b}-1\right)-\frac{\pi}{2}\right]\right\} \quad \text{.......... (14)}$$

Now setting up the total passive force $\int dE$ for $\alpha_w \geq 0$ or Zone II + Zone III we have,

$$\int dE = -\gamma\int_0^{x_0}\frac{y' - \tan\phi}{1 + y'\tan\phi}y\,dx\Bigg|_{\text{Zone II}} + 0.5\gamma\frac{1+\sin\phi}{1-\sin\phi}y_0^2\Bigg|_{\text{Zone III}}$$

$$= -\gamma\int_y^{y_0}\frac{y' - \tan\phi}{1 + y'\tan\phi}y x'\,dy\Bigg|_{\text{Zone II}} + 0.5\gamma\frac{1+\sin\phi}{1-\sin\phi}y_0^2\Bigg|_{\text{Zone III}} \quad \text{.......... (15)}$$

Substituting Eq. 11 in Eq. 15 and integrating yields,

$$E_p = \int dE = 0.5\gamma y^2\left(\frac{1+\sin^2\phi}{\cos^2\phi}\right) + \gamma\left(\frac{\tan\phi}{\cos\phi}\right)\left\{\frac{4y_0 y - 2y_b y + 2y_b y_0 - 3y_b^2}{4}\sqrt{\frac{y_b y - y^2}{y_b y_0 - y_0^2}}\right.$$
$$\left. + \frac{3}{4}y_b^2 + \frac{(3y_b - 2y_0)y_b^2}{8\sqrt{y_b y_0 - y_0^2}}\left[\sin^{-1}\left(\frac{2y}{y_b}-1\right) - \sin^{-1}\left(\frac{2y_0}{y_b}-1\right)\right]\right\} \quad \text{....... (16)}$$

Let $n = y_b / y$ and $m = y_0 / y_b$ and substitute in Eq. 16 and find K'p yields,

$$K'p = \frac{E_p}{0.5\gamma y^2} = \frac{1+\sin^2\phi}{\cos^2\phi} + 2\left(\frac{\tan\phi}{\cos\phi}\right)\left\{\frac{4mn - 2n + 2mn^2 - 3n^2}{4}\sqrt{\frac{n-1}{mn^2 - m^2n^2}} + \frac{3}{4}n^2 + \frac{(3n-2mn)n^2}{8\sqrt{mn^2 - m^2n^2}}\left[\sin^{-1}\left(\frac{2}{n}-1\right) - \sin^{-1}(2m-1)\right]\right\} \quad \text{............... (17)}$$

A relation between n and m can be found from Eq. 1 and Eq. 11 we have:

$$-\cot\alpha_w = -\tan\phi\left[1 + \frac{1}{\sin\phi}\sqrt{\frac{1-m}{m}\frac{1}{n-1}}\right] \quad \text{.. (18)}$$

Kp from Shields and Tolunay (1973)[2] can be found as

$$Kp = \frac{K'p}{1 - \tan\delta\tan(\alpha_w + \phi)} \quad \text{.. (19)}$$

If minimizing Eq. 19 with respect to n or m the result is $K'p = \tan^2(\pi/4 + \phi/2)$ and the slip surface becomes a line. Which is not physically acceptable and thus the result cannot be obtained from minimizations.

To match experiment and assuming geometrical harmony, a good approximation can be

$$m = 0.3\frac{\alpha_w}{\pi/4 - \phi/2} + 0.7 \quad \text{.. (20)}$$

Thus using Eq. 17, Eq. 18 and Eq, 20, Kp can be found from Eq. 19 (see **kp2.xls** for results)

Now setting up the total passive force $\int dE$ for $\alpha_w < 0$ or Zone I + Zone II + Zone III we have,

$$\int dE = \gamma\int_{y_b}^{y}\frac{\tan(-\alpha) - \tan\phi}{1 + \tan(-\alpha)\tan\phi}yx'dy\bigg|_{\text{Zone I}} - \gamma\int_{y_b}^{y_0}\frac{y' - \tan\phi}{1 + y'\tan\phi}yx'dy\bigg|_{\text{Zone II}} + 0.5\gamma\frac{1+\sin\phi}{1-\sin\phi}y_0^2\bigg|_{\text{Zone III}} \quad \text{......................... (21)}$$

Where, ϕ is replaced by $-\phi$ in Zone I, α is negative and y' is positive and $\tan\alpha = -y'$. Substituting $m = 0.7$ and $n = 1$ in Eq. 17 yields the Zone II and Zone III integrations in Eq. 21, and Eq. 21 becomes

$$\int dE = -\gamma \int_{y}^{y_b} \frac{y' - \tan\phi}{1 + y'\tan\phi} y x' dy \Bigg|_{\text{Zone I}} + 0.5\gamma y_b^2 \left[\frac{1 + \sin^2\phi}{\cos^2\phi} + 2.5119 \frac{\tan\phi}{\cos\phi} \right] \quad \text{……………………… (22)}$$

Substituting Eq. 12 in Eq. 22 and integrating yields,

$$E''_p = \int dE = 0.5\gamma y_b^2 \left[\frac{1 + \sin^2\phi}{\cos^2\phi} + 2.5119 \frac{\tan\phi}{\cos\phi} \right] - 0.5\gamma (y_b^2 - y^2) \left(\frac{1 + \sin^2\phi}{\cos^2\phi} \right)$$
$$+ \gamma \left(\frac{\tan\phi}{\cos\phi} \right) \left\{ -\frac{4 y_a y - 2 y_b y + 2 y_b y_a - 3 y_b^2}{4} \sqrt{\frac{y_b y - y^2}{y_b y_a - y_a^2}} \right. \quad \text{………………. (23)}$$
$$\left. + \frac{(3 y_b - 2 y_a) y_b^2}{8\sqrt{y_b y_a - y_a^2}} \left[\frac{\pi}{2} - \sin^{-1}\left(\frac{2y}{y_b} - 1 \right) \right] \right\}$$

Let $l = y_a / y_b$ and $n = y_b / y$ and substitute in Eq. 23 and find K”p yields,

$$K''p = \frac{E''_p}{0.5\gamma y^2} = n^2 \left[\frac{1 + \sin^2\phi}{\cos^2\phi} + 2.5119 \frac{\tan\phi}{\cos\phi} \right] - n^2 \left(1 - \frac{1}{n^2} \right) \left(\frac{1 + \sin^2\phi}{\cos^2\phi} \right)$$
$$+ 2n^2 \left(\frac{\tan\phi}{\cos\phi} \right) \left\{ -\frac{4l - 2 + 2nl - 3n}{4n} \frac{\sqrt{\frac{1}{n} - \frac{1}{n^2}}}{\sqrt{l - l^2}} \right. \quad \text{…….………… (24)}$$
$$\left. + \frac{3 - 2l}{8\sqrt{l - l^2}} \left[\frac{\pi}{2} - \sin^{-1}\left(\frac{2}{n} - 1 \right) \right] \right\}$$

A relation between n and l can be found from Eq. 2 and Eq. 12 we have:

$$-\cot\alpha_w = -\tan\phi \left[1 - \frac{1}{\sin\phi} \sqrt{\frac{1 - l}{l} \frac{1}{n - 1}} \right] \quad \text{……………………………………………….. (25)}$$

Kp from Shields and Tolunay (1973)[2] can be found as

$$Kp = \frac{K''p}{1 - \tan\delta \tan(\alpha_w + \phi)} \quad \text{…………………………………………………………... (26)}$$

To match experiment and assuming geometrical harmony, a good approximation can be

$$l = -0.008\frac{\alpha_w}{\pi/4+\phi/2}+0.714 \qquad (27)$$

Thus using Eq. 24, Eq. 25 and Eq, 27, Kp can be found from Eq. 26 (see **kp2.xls** for results)

Conclusion:
The exact slip surface for the method of slices with no friction between the slices is derived from the method of variation. The condition chosen with the variation method requires the resultant force of the wedge of Zone I and Zone II remain in the same direction, which is considered effective for the direction chosen is of the actual resultant of the actual Kp. Because this slip surface gives an absolute minimum and a lower bound, it underestimates Kp for smaller ϕ when compared with experiment. This is an indication that the friction on the wall for smaller ϕ dissipate in more than the first slice at the wall. We have selected the boundary condition to match experiment, since no minimization is possible and only geometric relations and harmony are considered. The proposed solution is a lower bound and can be used to obtain a Kp with an effective ϕ.

References:

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3- Terzaghi, Karl (1943). Theoretical Soil Mechanics, Wiley and Sons, New York, pp. 39, pp. 152-155, pp. 38-41, pp. 113-117, pp. 107 and pp. 121.